Study Material

CONTROL SYSTEM ENGINEERING
(As per SCTE&VT,Odisha new syllabus)

4th Semester Electronics & Telecom Engineering

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CHAPTER 1 :- FUNDAMENTAL OF CONTROL SYSTEM

Control system engineering is the branch of engineering which deals with the principles of control theory to design a system which gives desired behaviour in a controlled manner. Hence, this is interdisciplinary. **Control system** engineers analyze, design, and optimize complex systems which consist of highly integrated coordination of mechanical, electrical, chemical, metallurgical, electronic or pneumatic elements. Thus **control engineering** deals with diverse range of dynamic systems which include human and technological interfacing.

Control system engineering focuses on analysis and design of systems to improve the speed of response, accuracy and stability of system. The two methods of control system include classical methods and modern methods. The mathematical model of system is set up as first step followed by analysis, designing and testing. Necessary conditions for the stability are checked and finally optimization follows.

In classical method, mathematical modelling is usually done in time domain, frequency domain or complex s domain. Step response of a system is mathematically modelled in time domain differential analysis to find its settling time, % overshoot etc. Laplace transforms are most commonly used in frequency domain to find the open loop gain, phase margin, band width etc of system. Concept of transfer function, sampling of data, poles and zeros, system delays all comes under the **classical control engineering** stream.

Modern control engineering deals with Multiple Input Multiple Output (MIMO) systems, State space approach, Eigen values and vectors etc. Instead of transforming complex ordinary differential equations, modern approach converts higher order equations to first order differential equations and solved by vector method.

Automatic control systems are most commonly used as it does not involve manual control. The controlled variable is measured and compared with a specified value to obtain the desired result. As a result of automated systems for control purposes, the cost of energy or power as well as the cost of process will be reduced increasing its quality and productivity.

Before I introduce you the theory of control system it is very essential to know the various **types of control systems**. Now there are various types of systems, we are going to discuss only those types of systems that will help us to understand the theory of control system and detail description of these types of system are given below:

Linear Control Systems

In order to understand the **linear control system**, we should know the principle of superposition. The principle of superposition theorem includes two the important properties and they are explained below:

Homogeneity: A system is said to be homogeneous, if we multiply input with some constant 'A' then output will also be multiplied by the same value of constant (i.e. A).

Additivity: Suppose we have a system 'S' and we are giving the input to this system as 'a_1' for the first time and we are getting output as 'b_1' corresponding to input 'a_1'. On second time we are giving input 'a_2' and correspond to this we are getting output as 'b_2'. Now suppose this time we giving input as summation of the previous inputs (i.e. $a_1 + a_2$) and corresponding to this input suppose we are getting output as ($b_1 + b_2$) then we can say that system 'S' is following the property of additivity. Now we are able to define the **linear control systems** as those **types of control systems** which follow the principle of homogeneity and additivity.

Examples of Linear Control System

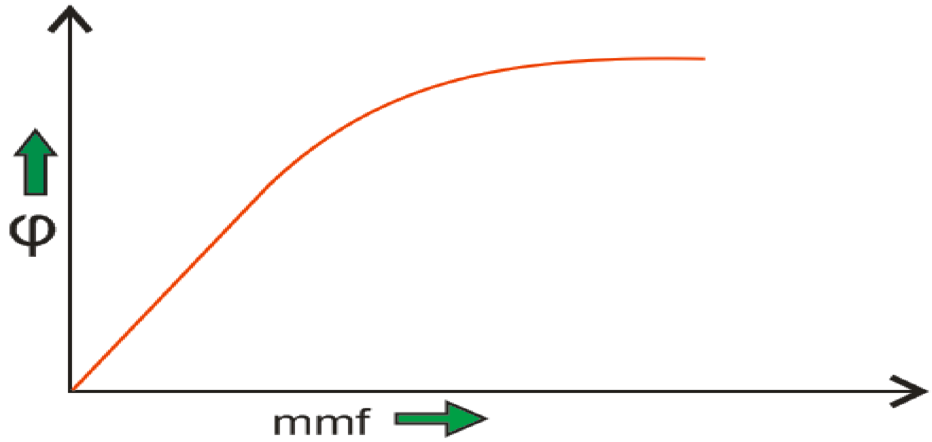
Consider a purely resistive network with a constant dc source. This circuit follows the principle of homogeneity and additivity. All the undesired effects are neglected and assuming ideal behaviour of each element in the network, we say that we will get linear voltage and current characteristic. This is the example of **linear control system**.

Non-linear Systems

We can simply define **non linear control system** as all those system which do not follow the principle of homogeneity. In practical life all the systems are non-linear system.

Examples of Non-linear System

A well known example of non-linear system is magnetization curve or no load curve of a dc machine. We will discuss briefly no load curve of dc machines here: No load curve gives us the relationship between the air gap flux and the field winding mmf. It is very clear from the curve given below that in the beginning there is a linear relationship between winding mmf and the air gap flux but after this, saturation has come which shows the non linear behavior of the curve or characteristics of the **non linear control system**.



In recent years, **control systems** plays main role in the development and advancement of modern technology and civilization. Practically every aspects of our day-to-day life is affected less or more by some control system. A bathroom toilet tank, a refrigerator, an air conditioner, a geezer, an automatic iron, an automobile all are control system. These systems are also used in industrial process for more output. We find control system in quality control of products, weapons system, transportation systems, power system, space technology, robotics and many

Requirement Of Good Control System

Accuracy: Accuracy is the measurement tolerance of the instrument and defines the limits of the errors made when the instrument is used in normal operating conditions. Accuracy can be improved by using feedback elements. To increase accuracy of any control system error detector should be present in control system.

Sensitivity: The parameters of control system are always changing with change in surrounding conditions, internal disturbance or any other parameters. This change can be expressed in terms of sensitivity. Any control system should be insensitive to such parameters but sensitive to input signals only.

Noise: An undesired input signal is known as noise. A good control system should be able to reduce the noise effect for better performance.

Stability: It is an important characteristic of control system. For the bounded input signal, the output must be bounded and if input is zero then output must be zero then such a control system is said to be stable system.

Bandwidth: An operating frequency range decides the bandwidth of control system. Bandwidth should be large as possible for frequency response of good control system.

Speed: It is the time taken by control system to achieve its stable output. A good control system possesses high speed. The transient period for such system is very small.

Oscillation: A small numbers of oscillation or constant oscillation of output tend to system to be stable.

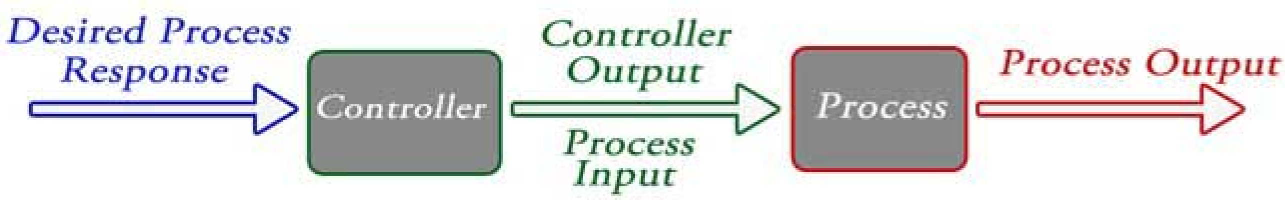
Types Of Control Systems

There are two main **types of control system**. They are as follow

1. **Open loop control system**
2. **Closed loop control system**

Open Loop Control System

A control system in which the control action is totally independent of output of the system then it is called **open loop control system**. Open loop system is also called as Manual control system. Fig – 1 shows the block diagram of open loop control system in which process output is totally independent of controller action.



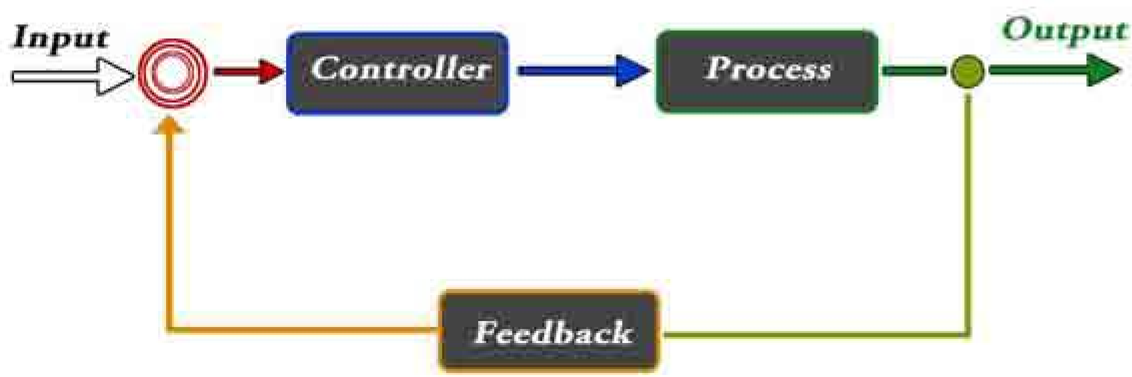
Practical Examples Of Open Loop Control System

1. **Electric Hand Drier** – Hot air (output) comes out as long as you keep your hand under the machine, irrespective of how much your hand is dried.
2. **Automatic Washing Machine** – This machine runs according to the pre-set time irrespective of washing is completed or not.
3. **Bread Toaster** – This machine runs as per adjusted time irrespective of toasting is completed or not.
4. **Automatic Tea/Coffee Maker** – These machines also function for pre adjusted time only.
5. **Timer Based Clothes Drier** – This machine dries wet clothes for pre – adjusted time, it does not matter how much the clothes are dried.
6. **Light Switch** – lamps glow whenever light switch is on irrespective of light is required or not.
7. **Volume on Stereo System** – Volume is adjusted manually irrespective of output volume level.
8. Advantages Of Open Loop Control System

1. Simple in construction and design.
2. Economical. Easy to maintain.

Closed Loop Control System

Control system in which the output has an effect on the input quantity in such a manner that the input quantity will adjust itself based on the output generated is called **closed loop control system**. Open loop control system can be converted in to closed loop control system by providing a feedback. This feedback automatically makes the suitable changes in the output due to external disturbance. In this way closed loop control system is called automatic control system. Figure below shows the block diagram of closed loop control system in which feedback is taken from output and fed in to input.



Practical Examples Of Closed Loop Control System

1. **Automatic Electric Iron** – Heating elements are controlled by output temperature of the iron.
2. **Servo Voltage Stabilizer** – Voltage controller operates depending upon outputvoltage of the system.
3. **Water Level Controller**– Input water is controlled by water level of the reservoir.
4. **Missile Launched & Auto Tracked by Radar** – The direction of missile is controlled by comparing the target and position of the missile.
5. **An Air Conditioner** – An air conditioner functions depending upon the temperature of the room.
6. **Cooling System in Car** – It operates depending upon the temperature which it controls.

Advantages OF Closed Loop Control System

1. Closed loop control systems are more accurate even in the presence of non-linearity.
2. Highly accurate as any error arising is corrected due to presence of feedback signal.
3. Bandwidth range is large.
4. Facilitates automation.
5. The sensitivity of system may be made small to make system more stable.
6. This system is less affected by noise.

Disadvantages Of Closed Loop Control System

7. They are costlier.
8. They are complicated to design.
9. Required more maintenance.
10. Feedback leads to oscillatory response.
11. Overall gain is reduced due to presence of feedback.
12. Stability is the major problem and more care is needed to design a stable closed loop system.

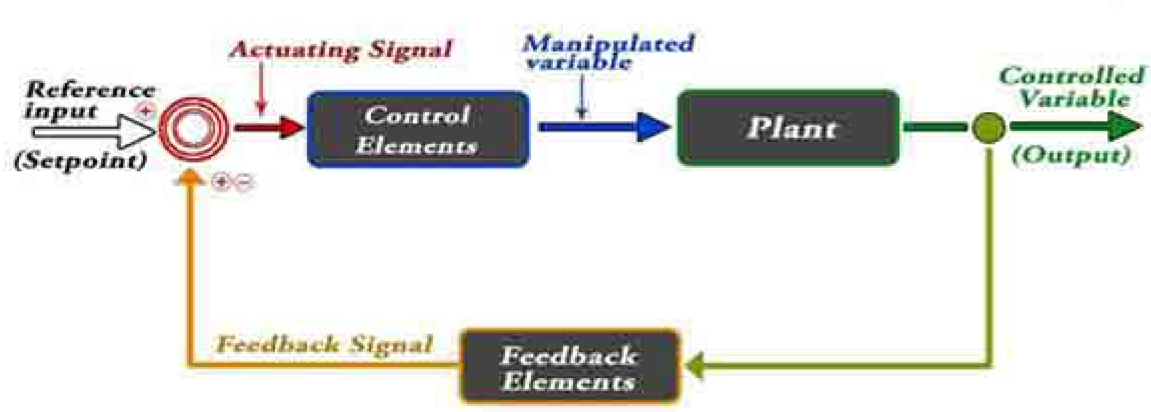
Comparison of Closed Loop And Open Loop Control System

OPEN LOOP CONTROL SYSTEM	CLOSED LOOP CONTROL SYSTEM
The feedback element is absent.	The feedback element is always present.
An error detector is not present.	An error detector is always present
It is stable one.	It may become unstable.
Easy to construct.	Complicated construction.
It is an economical.	It is costly.
Having small bandwidth.	Having large bandwidth
It is inaccurate.	It is accurate.
Examples: Hand drier, tea Maker	Examples:Servo voltage stabilizer,

Feedback Loop Of Control System

A feedback is a common and powerful tool when designing a control system. Feedback loop is the tool which take the system output into consideration and enables the system to adjust its performance to meet a desired result of system.

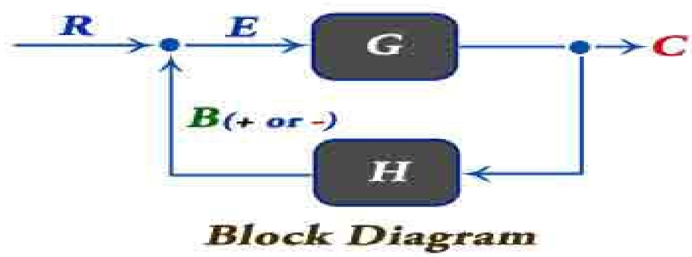
In any control system, output is affected due to change in environmental condition or any kind of disturbance. So one signal is taken from output and is fed back to the input. This signal is compared with reference input and then error signal is generated. This error signal is applied to controller and output is corrected. Such a system is called feedback system. Figure below shows the block diagram of feedback system.



When feedback signal is positive then system called positive feedback system. For positive feedback system, the error signal is the addition of reference input signal and feedback signal. When feedback signal is negative then system is called negative feedback system. For negative feedback system, the error signal is given by difference of reference input signal and feedback signal.

Effect Of Feedback

Refer figure beside, which represents feedback system where
R = Input signal
E = Error signal
G = forward path gain
H = Feedback   C = Output signal   B=Feedback signal



1. Error between system input and system output is reduced.

2. System gain is reduced by a factor 1/(1±GH).

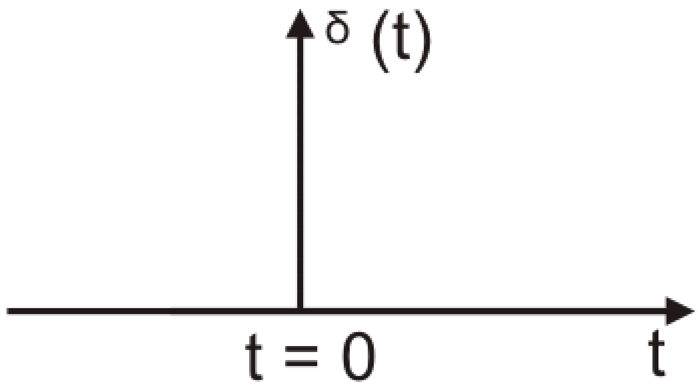
3 Improvement in sensitivity.

4. Stability may be affected.

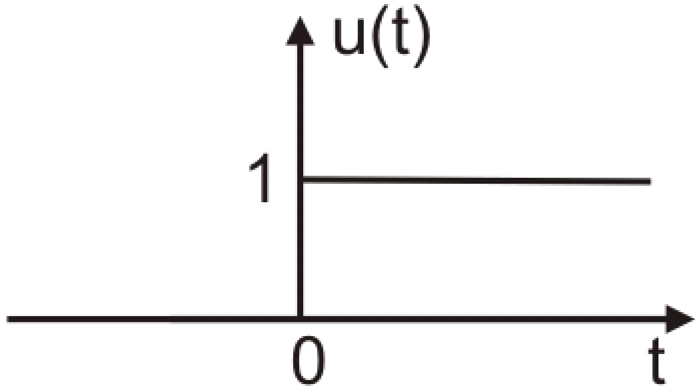
5. Improve the speed of response.

Standard Input Test Signals : These are also known as test input signals. The input signal is very complex in nature, it is complex because it may be a combination of various other signals. Thus it is very difficult to analyze characteristic performance of any system by applying these signals. So we use test signals or standard input signals which are very easy to deal with. We can easily analyze the characteristic performance of any system more easily as compared to non standard input signals. Now there are various types of standard input signals and they are written below:

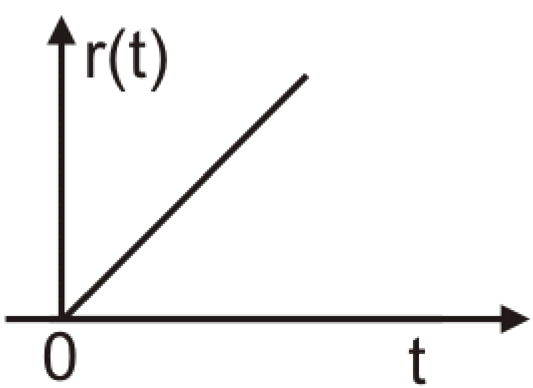
Unit Impulse Signal : In the time domain it is represented by ∂(t). The Laplace transformation of unit impulse function is 1 and the corresponding waveform associated with the unit impulse function is shown below.



Unit Step Signal : In the time domain it is represented by u (t). The Laplace transformation of unit step function is 1/s and the corresponding waveform associated with the unit step function is shown below.

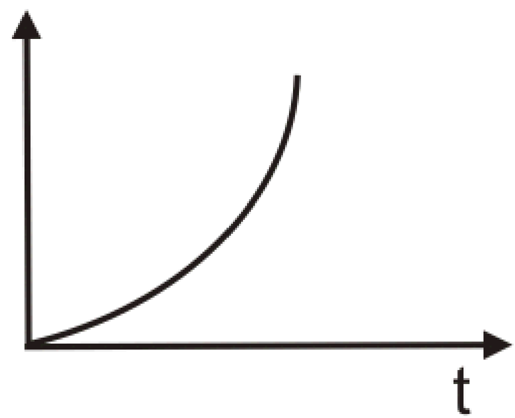


Unit Ramp signal : In the time domain it is represented by r (t). The Laplace transformation of unit ramp function is $1/s^2$ and the corresponding waveform associated with the unit ramp function is shown below.



Unit Ramp Signal

Parabolic Type Signal : In the time domain it is represented by $t^2 / 2$. The Laplace transformation of parabolic type of the function is $1 / s^3$ and the corresponding waveform associated with the parabolic type of the function is shown below.



CHAPTER 2

TRANSFER FUNCTION

TRANSFER FUNCTION

For any control system there exists a reference input termed as excitation or cause which operates through a transfer operation termed as **transfer function** and produces an effect resulting in controlled output or response. Thus the cause and effect relationship between the output and input is related to each other through a **transfer function**.

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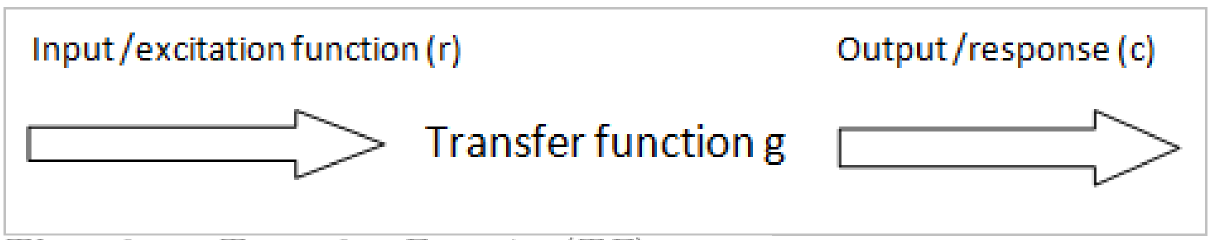
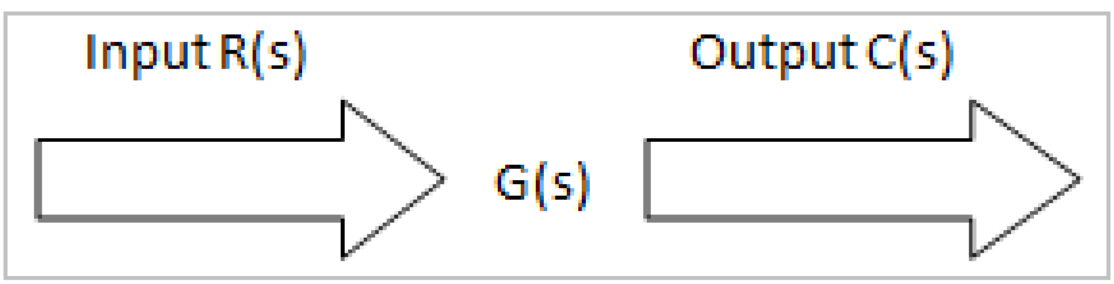


$$Therefore,\ Transfer\ Function(TF)\ g\ =\ \frac{c}{r}$$

It is not necessary that the output will be of same category as that of the input. For example – in case of an electrical motor, the input is an electrical quantity and output is a mechanical one. In control system all mathematical functions are converted to their corresponding Laplace transforms. So the transfer function is expressed as a ratio of Laplace transform of output function to Laplace transform of input function.



The transfer function can be expressed as

$$G(s)\ =\ \frac{C(s)}{R(s)}$$

While doing Laplace transform, while determining transfer function we assume all initial conditions to be zero.

$$Hence,\ transfer\ function\ G(s)\ =\ \frac{Lc(t)}{Lr(t)}$$

The transfer function of a control system is defined as the ration of the Laplace transform of the output variable to Laplace transform of the input variable assuming all initial conditions to be zero.

Procedure for determining the transfer function of a control system are as follows :

1. First deduce the equations for the system
2. Now take Laplace transform of the system equations, assuming initial conditions as zero.
3. Specify system output and input
4. Lastly take the ratio of the Laplace transform of the output and the Laplace transform of the input which is **the required transfer function**

Methods of obtaining a Transfer function: There are major two ways of obtaining a transfer function for the control system .The ways are –

• Block diagram method : It is not convenient to derive a complete transfer function for a complex control system. Therefore the transfer function of each element of a control system is represented by a block diagram. Block diagram reduction techniques are applied to obtain the desired transfer function.

Signal Flow graphs: Signal Flow Graph is a modified form of a block diagram which gives a pictorial representation of a control system . Signal flow graph further shortens the representation of a control system.

The transfer function of a system is completely specified in terms of its poles and zeroes and the gain factor. Let us know about the poles and zeroes of a transfer function in brief.

$$G(s) = \frac{C(s)}{R(s)} = K\ (system\ gain)$$

Where, K = system gain,

z_1 , z_2 , ………………… z_m = zero's of the transfer function

p_1 , p_2 , ………………… p_n = pole's of the transfer function

Putting the denominator of equation (i) equal to zero we get the poles value of the transfer function. For this the T.F is infinity.

Putting the numerator of equation (ii) equal to zero we get the value of zero of the transfer function. For this T.F is equal to zero.

There are two types of transfer functions :-

i) Open loop transfer function(O.L.T.F) : Transfer function of the system without feedback path or loop.

ii) Closed loop transfer function (C.L.T.F) : Transfer function of the system with feedback path or loop.

EXAMPLE 1.1. Find the transfer function of the given network

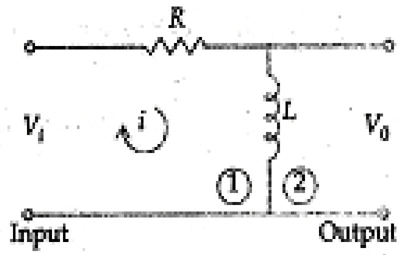


Fig. 1.9.

Solution : Step 1 : Apply KVL in mesh (1)

$$V_i = Ri + L\frac{di}{dt} \qquad ...(1.13)$$

Apply *KVL* in mesh (2)

$$V_0 = L\frac{di}{dt} \qquad ...(1.14)$$

Step 2 : Take Laplace transform of equations (1.13) and (1.14) with assumption that all initial conditions are zero.

$$V_i(s) = RI(s) + sLI(s) \qquad ...(1.15)$$

$$V_0(s) = sLI(s) \qquad ...(1.16)$$

Step 3 : Calculation of transfer function

$$\frac{V_0(s)}{V_i(s)} = \frac{sLI(s)}{(R+sL)I(s)}$$

$$\frac{V_0(s)}{V_i(s)} = \frac{sL}{R+sL} \qquad ...(1.17)$$

Equation 1.17 is the required transfer function.

EXAMPLE 1.2. Determine the transfer function of the electrical network shown in Fig. 1.10.

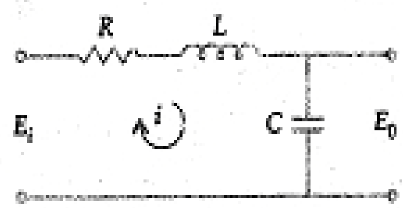


Fig. 1.10.

Solution : Step 1 : Apply KVL in both meshes

$$E_i = Ri + L\frac{di}{dt} + \frac{1}{C}\int idt \qquad ...(1.18)$$

$$E_0 = \frac{1}{C}\int idt \qquad ...(1.19)$$

Step 2 : Take Laplace transform of equations (1.18) and (1.19)

$$E_i(s) = RI(s) + sLI(s) + \frac{1}{Cs}I(s) = I(s)\left[R + sL + \frac{1}{Cs}\right]$$

$$E_i(s) = I(s)\left[\frac{RCs + s^2LC + 1}{Cs}\right] \qquad ...(1.20)$$

$$E_0(s) = \frac{1}{Cs}I(s) \qquad ...(1.21)$$

Step 3 : Determination of transfer function

$$\frac{E_0(s)}{E_i(s)} = \frac{I(s)}{Cs}\cdot\frac{Cs}{I(s)\left[s^2Lc + SRC + 1\right]}$$

$$\frac{E_0(s)}{E_i(s)} = \frac{1}{S^2LC + SRC + 1} \qquad \text{Ans.} \qquad ...(1.22)$$

EXAMPLE 1.3. Obtain the transfer function $\frac{V_2(s)}{V_1(s)}$ for Fig. 1.11.

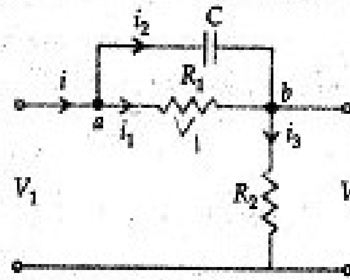


Fig. 1.11.

Solution : Step 1 : KCL at node '*a*'

$$i = i_1 + i_2 \qquad \text{...(1.23)}$$

$$i_1 = \frac{V_1 - V_2}{R_1}$$

$$i_2 = C\frac{d}{dt}(V_1 - V_2)$$

$$i = i_3 = \frac{V_2}{R_2}$$

Put all these values in equation (1.23)

$$\frac{V_2}{R_2} = \frac{V_1 - V_2}{R_1} + C\frac{d}{dt}(V_1 - V_2) \qquad \text{...(1.24)}$$

Step 2 : Take Laplace transform of equation (1.24)

$$\frac{V_2(s)}{R_2} = \frac{1}{R_1}V_1(s) - \frac{1}{R_1}V_2(s) + Cs\,V_1(s) - Cs\,V_2(s)$$

$$\frac{V_2(s)}{R_2} + \frac{1}{R_1}V_2(S) + Cs\,V_2(s) = \frac{1}{R_1}V_1(s) + Cs\,V_1(s)$$

$$V_2(s)\left[\frac{1}{R_1} + \frac{1}{R_2} + Cs\right] = V_1(s)\left[\frac{1}{R_1} + Cs\right]$$

Step 3 : Determination of transfer function

$$V_2(s)\left[\frac{R_1 + R_2 + R_1R_2Cs}{R_1R_2}\right] = V_1(s)\left[\frac{1 + R_1Cs}{R_1}\right]$$

$$\frac{V_2(s)}{V_1(s)} = \frac{R_2 + R_1R_2Cs}{R_1 + R_2 + R_1R_2Cs} \qquad \text{Ans.} \qquad \text{...(1.25)}$$

EXAMPLE 1.4. Find the transfer function of lag network shown in Fig. 1.12.

Solution : Step 1 : Apply KVL in both meshes

$$e_i(t) = R_1 i(t) + R_2 i(t) + \frac{1}{C}\int i(t)\,dt \qquad \text{...(1.26)}$$

$$e_0(t) = R_2 i(i) + \frac{1}{C}\int i(t)\,dt \qquad \text{...(1.27)}$$

Step 2 : Laplace transform of equation (1.26) and (1.27)

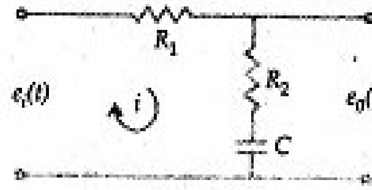


Fig. 1.12

$$E_i(s) = \left[R_1 + R_2 + \frac{1}{Cs}\right]I(s)$$

$$E_0(s) = \left[R_2 + \frac{1}{Cs}\right]I(s)$$

Step 3 : Calculation of transfer function

$$E_0(s) = \frac{\left[R_2 + \frac{1}{Cs}\right] I(s)}{\left[\frac{R_1Cs + R_2Cs + 1}{Cs}\right] I(s)}$$

$$\frac{E_0(s)}{E_i(s)} = \frac{1 + R_2Cs}{1 + R_1Cs + R_2Cs} \qquad ...(1.28)$$

Equation (1.28) is the required transfer function.

EXAMPLE 1.5. Determine the transfer function of Fig. 1.13.

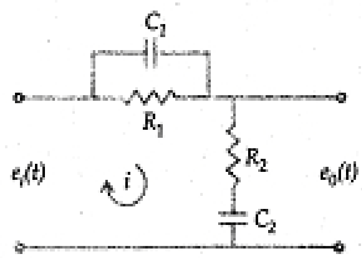


Fig. 1.13.

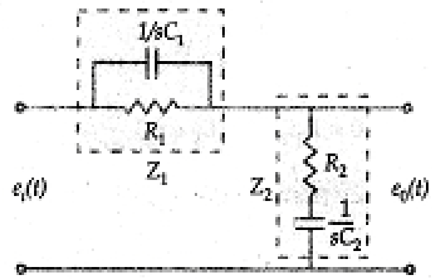


Fig. 1.14.

Solution : Step 1 : calculation of Z_1 :

$$Z_1 = \frac{R_1 \frac{1}{sC_1}}{R_1 + \frac{1}{sC_1}} = \frac{R_1}{R_1C_1s + 1} \qquad ...(1.29)$$

Step 2 : Calculation of Z_2 :

$$Z_2 = R_2 + \frac{1}{sC_2} = \frac{R_2C_2S + 1}{sC_2} \qquad ...(1.30)$$

Step 3 : Calculation of transfer function in terms of Z_1 and Z_2

$$\frac{E_0(s)}{E_i(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} \qquad ...(1.31)$$

Step 4 : Calculation of transfer function in terms of R_1, R_2, C_1 and C_2. Put the values of $Z_1(s)$ and $Z_2(s)$ from equations (1.29) & 1.30 in equation (1.31)

$$\frac{E_0(s)}{E_i(s)} = \frac{(1 + R_2C_2S)/Sc_2}{\frac{R_1}{C_1R_1S + 1} + \frac{R_2C_2S + 1}{SC_2}}$$

$$\frac{E_0(s)}{E_i(s)} = \frac{(1 + R_1C_1S)(1 + R_2C_2S)}{(1 + R_1C_1S)(1 + R_2C_2S) + R_1C_2S} \qquad ...(1.32)$$

The above equation is the required transfer function of the given circuit.

Chapter3

CONTROL SYSTEM COMPONENTS

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MATHAMATICAL MODELING OF PHYSICAL SYSTEM

In a control system, the devices which are used to convert the process variables in one form to another form is known as Transducer. Transducer can also be defined as a device which transforms the energy from one form to another. For example a thermocouple converts the heat energy to electrical voltage. In control system the following devices are used as a transducer

1. **Potentiometer**
2. **DC Servomotor**
3. **AC servomotor**
4. **Synchros**
5. **Stepper Motor**
6. **Magnetic Amplifier**
7. **Tachogenerator**
8. **Gyroscope**
9. **Differential Transformer**

<u>Potentiometer</u> - A Potentiometer is a simple device which is used for mechanical displacement either linear or angular. Thus a Potentiometer is electro mechanical transducer which converts the mechanical energy to electrical energy. The input to the device is in the form of linear mechanical displacement or rotational mechanical displacement. When the voltage is applied across the fixed terminal. The output voltage is proportional to the displacement. Let Ei=Input voltage Eo=Output voltage

x_i = displacement from zero position

x_t = total length of translational potentiometer

R = total resistance of potentiometer

Under ideal condition the output voltage E_0 is given by

$$E_0 = \frac{x_i}{x_t} E_i \quad \text{...(9.1)}$$

Equation (9.1) shows a linear relationship shown in Fig. 9.2.

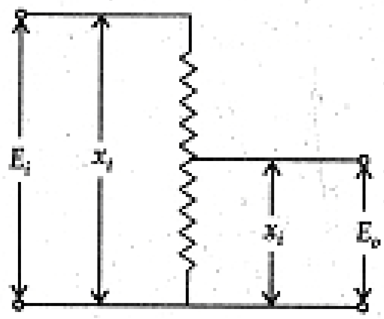


Fig. 9.1.

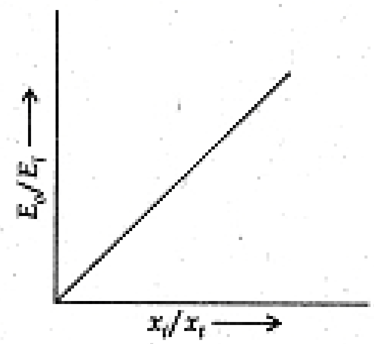


Fig. 9.2

Similarly for rotational motion, the output voltage E_0 is given by

$$E_0 = E_i \cdot \frac{\theta_i}{\theta_t} \quad \text{...(9.2)}$$

where θ_i = input angular displacement (degree or radians)

θ_t = total travel of wiper (degree or radians)

Figure 9.3(*a*) Shows an arrangement of error sensing transducer. In this arrangement two potentiometers are connected in parallel. The output voltage taken across the variable terminals of the two potentiometer. The output voltage E_0 is given by

$$E_o = K(\theta_1 - \theta_2) \quad \text{...(9.3)}$$

E_i is the voltage applied, θ_1 and θ_2 are the angular displacement of the wiper, K is constant and is known as sensitivity. The block diagram is shown in Fig. 9.3 (*b*).

(*a*) (*b*)

Fig. 9.3.

9.3. SERVOMOTORS

Servomotors are used in feedback control systems. Servomotors have low rotor inertia and high speed of response. The servomotors are also know as control motors. The servomotors which are used in feedback control system should have linear relationship between electrical control signal and rotor speed, torque speed characteristic should be linear, the response of the servomotor should be fast and inertia should be low.

9.4. TYPES OF SERVOMOTORS

The servomotors are classified as

(*i*) A.C. servomotors

(*ii*) D.C. servomotors

(*iii*) Special servomotors

D.C. servomotors are further classified as armature controlled d.c. servomotors and field control d.c. servomotors.

9.4.1. A.C. Servomotors

These motors having two parts namely stator and rotor. A.C. servomotors are two phase induction motor. The stator has two distributed windings. These windings are displaced from each other by 90° electrical. One winding is called main winding or reference winding. The reference winding is excited by constant a.c. voltage. The other winding is called *control winding*. This winding is excited by variable control voltage of the same frequency as the reference winding, but having a phase displacement of 90° electrical. The variable control voltage for control winding is obtained from a servoamplifier. The direction of rotation depends upon phase relationship of voltages applied to the two windings. The direction of rotation of the rotor can be reversed by reversing the phase difference between control voltage and reference voltage.

The rotor of a.c. servomotors are of two types (*a*) squirrel cage rotor (*b*) drag cup type rotor. The squirrel cage rotor having large length and small diameter, so its resistance is very high. The air gap of squirrel cage is kept small. In drag cup type there are two airgaps. For the rotor a cup of non-magnetic conducting material is used. A stationary iron core is placed between the conducting cup to complete the magnetic circuit. The resistance of drag cup type is high and therefore having high starting torque. Generally aluminium is used for cup. Figure 9.4. Shows the schematic diagram of two phase a.c. servomotor and Fig. 9.5(*a*), and (*b*) shows the two types of rotor.

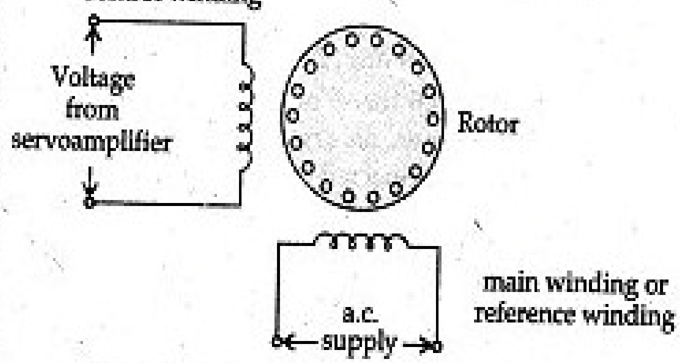


Fig. 9.4. A.C. servomotor

Fig. 9.5 (a) Squirrel cage rotor

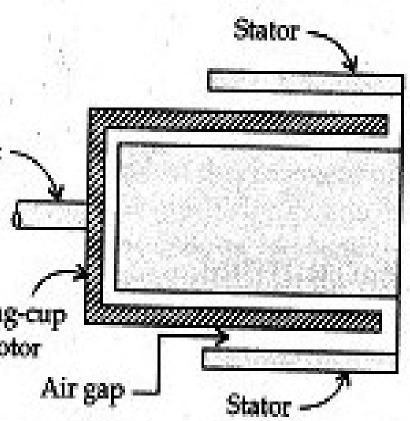


Fig. 9.5 (b) Drag cup type rotor

9.4.2. Torque-speed Characteristic

The torque speed characteristic of two phase induction motor depends upon the ratio of reactance to resistance. For high resistance and low reactance, the characteristic is linear and for large ratio of X to R it becomes non-linear as shown in Fig. 9.6(*a*). The torque-speed characteristics for various control voltages are almost linear as shown in Fig. 9.6(*b*).

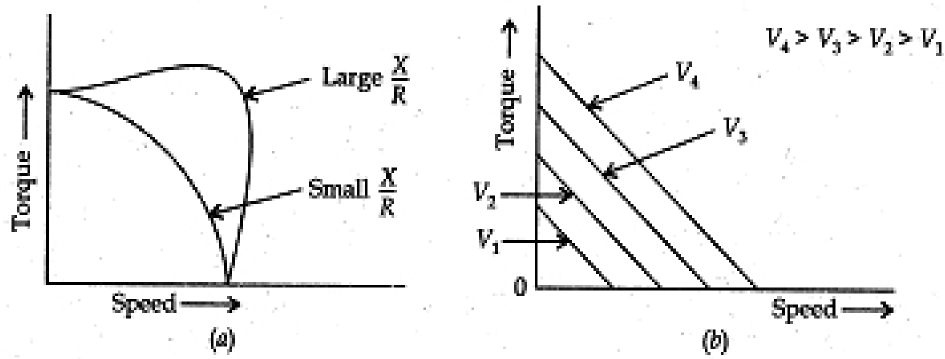
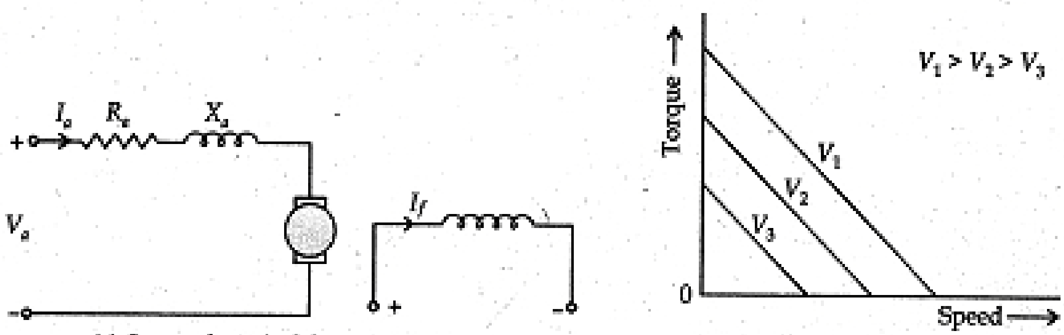


Fig. 9.6.

9.4.3. D.C. Servomotors

D.C. servomotors are separately excited or permanent magnet d.c. servomotors. The armature of d.c. servomotor has a large resistance, therefore torque speed characteristic is linear. The torque speed characteristic shows in Fig. 9.7(*b*). Fig 9.7(*a*) shows the schematic diagram of separately excited d.c. servomotor.



(*a*) *Separately excited d.c. servometer* (*b*) *Torque speed characteristic*

Fig. 9.7.

The d.c. servomotors can be controlled from armature side or from field. In field controlled d.c. servomotors the ratio of L/R is large *i.e.*, the time constant for field circuit is large. Due to large time constant, the response is slow and therefore they are not commonly used. Transfer function of field controlled d.c. servomotor is given in Chapter 1. The speed of the motor can be controlled by adjusting the voltage applied to the armature. In armature controlled d.c. servomotor the time constant is small and hence the response is fast. The efficiency is better than the field controlled motor. The transfer function of armature controlled d.c. servomotor is derived in Chapter 1.

9.4.4. Application of Servomotors

Servomotors are widely used in radars, electromechanical actuators, computers, machine tools, tracking and guidance system, process controllers and robots.

number of stacks or phases, then tooth pitch is given by $360°/T_r$ and angular displacement or step angle is given by $360°/nTr$. For exmaple 12 pole rotor, the pitch is 360/12 = 30 and the step angle will be 360/3 × 12 = 10° *i.e.*, rotor poles are displaced from each other by 10°.

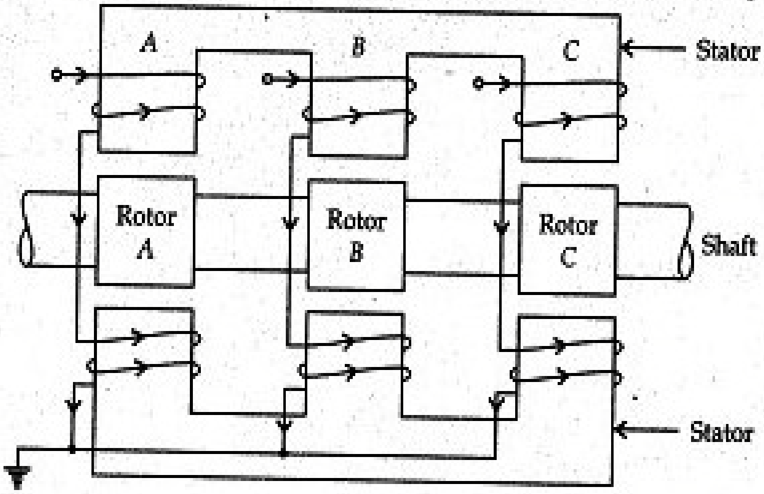


Fig. 9.10. Multi stack variable reluctance stepper motor

9.7. SYNCHROS

A synchro is an electromagnetic transducer which converts the angular position of a shaft into an electric signal. Synchros are used as detectors and encoders.

9.7.1. Synchro Transmitter

The construction of synchrotransmitter is very similar to that of a three phase alternator. The stator is made of laminated silicon steel and carries three phase star connected windings. The rotor is a rotating part, dumbbell shaped magnet with a single winding.

A single phase a.c. voltage is applied to the rotor through slip rings. Let applied a.c. voltage to the rotor is

$$e_r = E_r \sin \omega_0 t \qquad ...(9.7)$$

due to this applied voltage a magnetizing current will flow in the rotor coil. This magnetizing current produces sinusoidally varying flux and distributed in the air gap. Because of transformer action voltages get induced in all stator coil which is proportional to cosine of angle between stator and rotor coil axes.

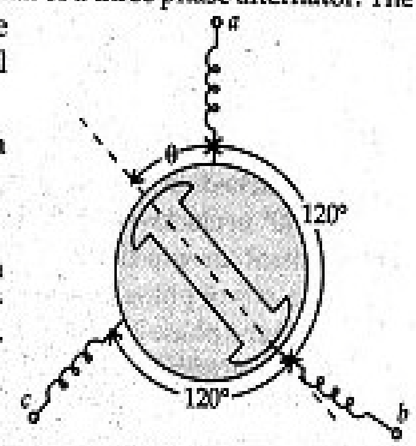


Fig. 9.11. Schematic diagram of synchro transmitter

Now, consider the rotor of synchro transmitter is at an angle θ, then voltages ineach stator coil with respect to neutral are

$$E_{an} = KE_r \sin \omega_0 t \cos \theta \qquad ....(9.8)$$

$$E_{bn} = KE_r \text{ son } \omega_0 t \cos (\theta + 120°) \qquad ....(9.9)$$

$$E_{cn} = KE_r \sin \omega_0 t \cos (\theta + 240°) \qquad ...(9.10)$$

Magnitudes of stator terminal voltages are

$$E_{cb} = E_{cn} - E_{bn}$$

$$= KE_r \sin \omega_0 t\, [\cos (\theta + 240°) - \cos (\theta + 120°)] = KE_r \sin \omega_0 t \left[\sqrt{3} \sin \theta\right]$$

$$\therefore \quad E_{cb} = \sqrt{3}\, KE_r \sin \omega_0 t \sin \theta \qquad ...(9.11)$$

Similarly, $E_{ac} = \sqrt{3}\ KE_r \sin\omega_0 t \sin(\theta + 120°)$ ...(9.12)

$E_{ba} = \sqrt{3}\ KE_r \sin\omega_0 t \sin(\theta + 240°)$ ...(9.13)

When $\theta = 0$, the maximum induced voltage will be E_{an} and E_{cb} will be zero. This position of the rotor is defined as electrical zero of the transmitter and is used as the reference for indicating the angular position of the rotor.

Thus, the input to the synchro transmitter is the angular position of the rotor shaft and the output are the three single phase voltages which are the function of the shaft position.

9.7.2. Synchro Control Transformer

Principle of operation of synchro control transformer is same as that of synchro transmitter. Rotor of snchro control transformer is cylindrical type. Synchro control transformer is an electromechanical device. The combination of synchro transmitter and synchro control transformer is used as an error detector. The function of error detector is to convert the difference of two shaft positions into an electrical signal. The Fig. 9.12, shows schematic diagram of synchro error detector.

The output of synchro transmitter is connected to the stator winding of the synchro control transformer. Therefore the same current will flow in the stator windings of synchro control transformer but in opposite direction. The voltage across the rotor terminals of control transformer is

$$e(t) = K_1\ V_r \cos\phi \sin\omega_0 t \quad ...(9.14)$$

where ϕ = angular displacement between the two rotors. When the two rotors are at an angle 90°, the voltage induced in control trnsformer is zero. This position is known as electrical zero position control transformer.

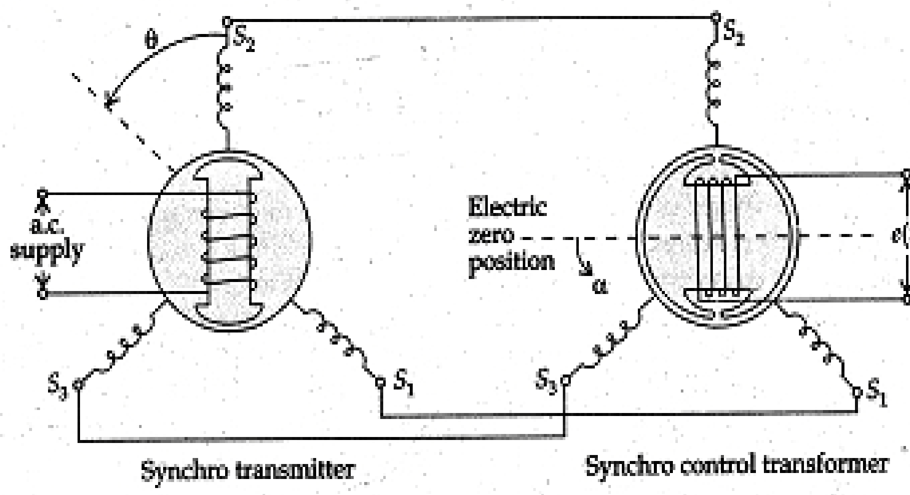


Fig. 9.12. Synchro error detector

Let the transmitter rotate through an angle 'θ' in the direction indicated and let control transformer rotor rotates in the same direction through an angle 'α'. Then

$$\phi = (90° - \theta + \alpha) \quad ...(9.15)$$

Put the value of ϕ in equation 9.14, we get

$$e(t) = K_1\ V_r \sin(\theta - \alpha) \sin\omega_0 t \quad ...(9.16)$$

From equation (9.16) it is clear that when two rotor shafts are not in alignment, the rotor voltage of control transformer is approximately a sine function of the difference between the two shaft angles.

For small angular displacement between two rotor position

$$e(t) = K_1\ V_r (\theta - \alpha) \sin\omega_0 t \quad ...(9.17)$$

CHAPTER- 4

Block Diagram & Signal Flow Graphs(SFG)

BLOCK-DIAGRAM REDUCTION
10.1 INTRODUCTION

Block Diagram: Pictorial representation of functions performed by each component of a system and that of flow of signals.

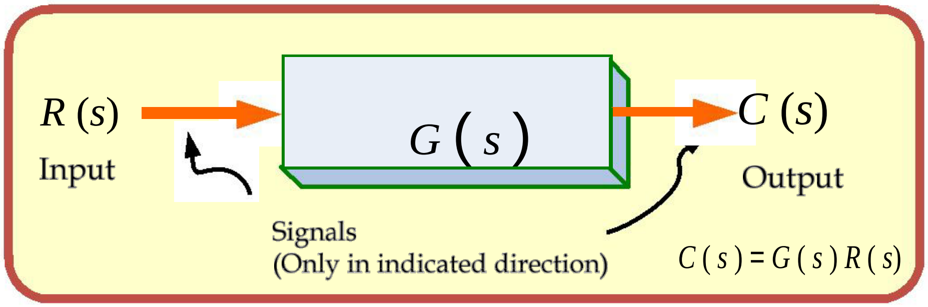


Figure Single block diagram representation.

Components for Linear Time Invariant System(LTIS):

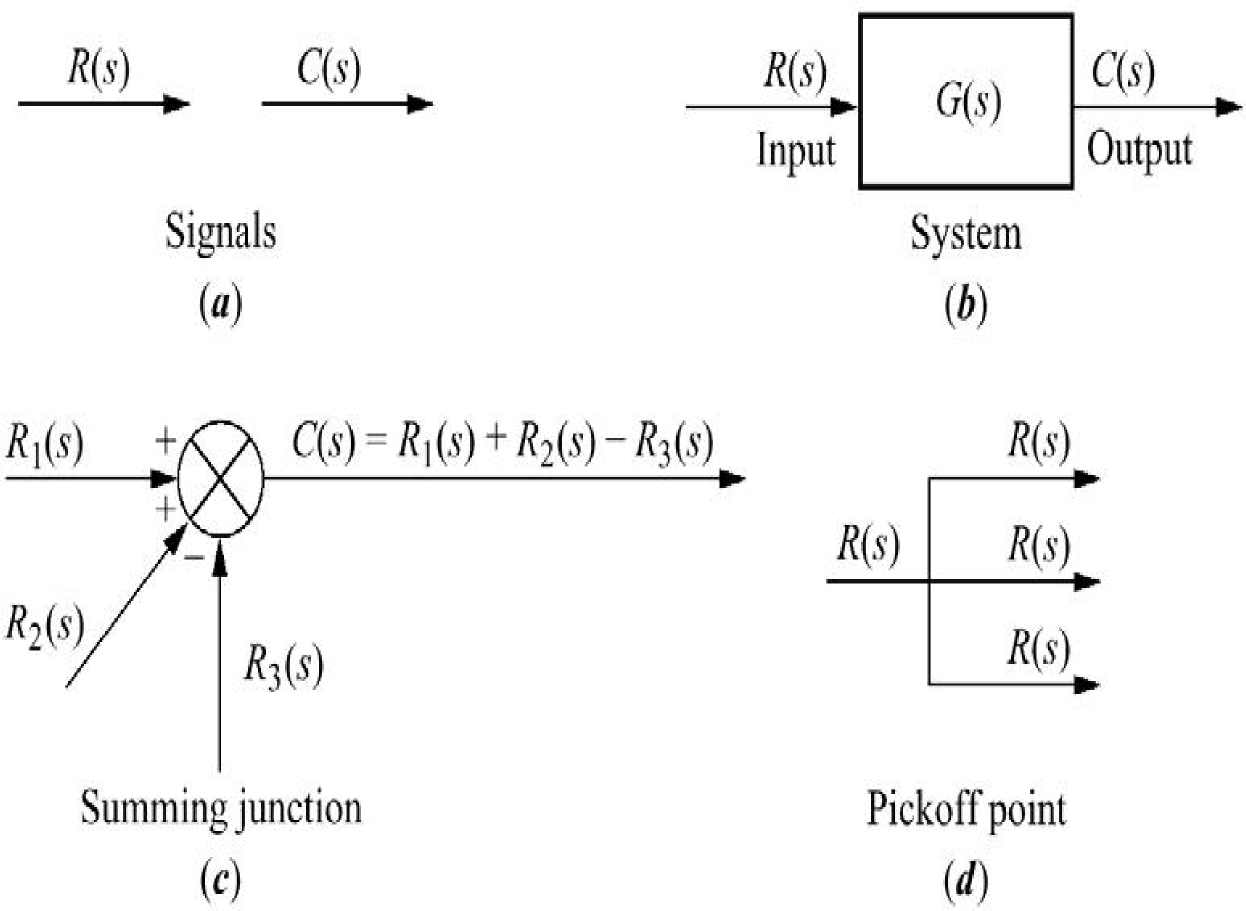


Figure Components for Linear Time Invariant Systems (LTIS).

Terminology:

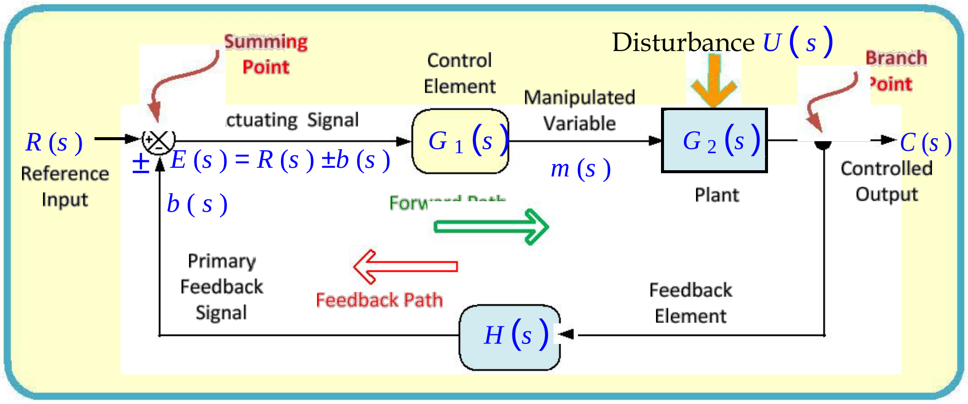


Figure Block Diagram Components.

1. ***Plant:*** A physical object to be controlled. The Plant $G_2(s)$, is the controlled system, of which a particular quantity or condition is to be controlled.
2. ***Feedback Control System (Closed-loop Control System)*** : A system which compares output to some reference input and keeps output as close as possible to this reference.
3. ***Open-loop Control System***: Output of the system is not feedback to the system.
4. ***Control Element*** $G_1(s)$, also called the ***controller***, are the components required to generate the appropriate control signal $M(s)$ applied to the plant.
5. ***Feedback Element*** $H(s)$ is the component required to establish the functional relationship between the primary feedback signal $B(s)$ and the controlled output $C(s)$.
6. ***Reference Input*** $R(s)$ is an external signal applied to a feedback control system in order to command a specified action of the plant. It often represents ideal plant output behavior.
7. The ***Controlled Output*** $C(s)$ is that quantity or condition of the plant which is controlled.
8. ***Actuating Signal*** $E(s)$, also called the error or control action, is the algebraic sum consisting of the reference input $R(s)$ plus or minus (usually minus) the primary feedback $B(s)$.
9. ***Manipulated Variable*** $M(s)$ (control signal) is that quantity or condition which the control elements $G_1(s)$ apply to the plant $G_2(s)$.
10. ***Disturbance*** $U(s)$ is an undesired input signal which affects the value of the controlled output $C(s)$. It may enter the plant by summation with $M(s)$, or via an intermediate point, as shown in the block diagram of the figure above.
11. ***Forward Path*** is the transmission path from the actuating signal $E(s)$ to the output $C(s)$.

12. ***Feedback Path*** is the transmission path from the output $C(s)$ to the feedback signal $B(s)$.

13. ***Summing Point***: A circle with a cross is the symbol that indicates a summing point. The $(+)$ or $(-)$ sign at each arrowhead indicates whether that signal is to be added or subtracted.

14. ***Branch Point:*** A branch point is a point from which the signal from a block goes concurrently to other blocks or summing points.

Definitions

- $G(s)$ ≡Direct transfer function = Forward transfer function.
- $H(s)$ ≡Feedback transfer function.
- $G(s)H(s)$ ≡ Open-loop transfer function.
- $C(s)/R(s)$ ≡ Closed-loop transfer function = Control ratio
- $C(s)/E(s)$≡ Feed-forward transfer function.

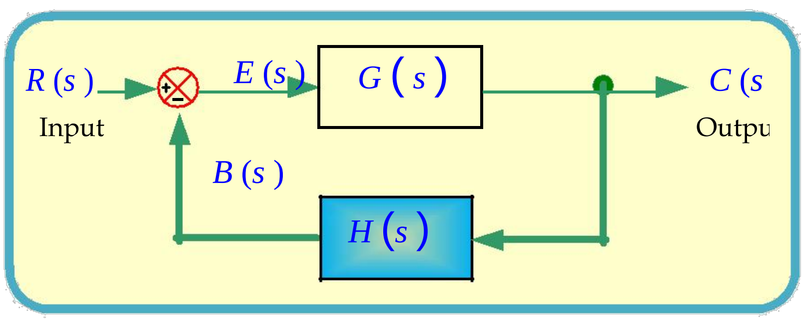


Figure Block diagram of a closed-loop system with a feedback element.

10.2 BLOCK DIAGRAMS AND THEIR SIMPLIFICATION

Cascade (Series) Connections

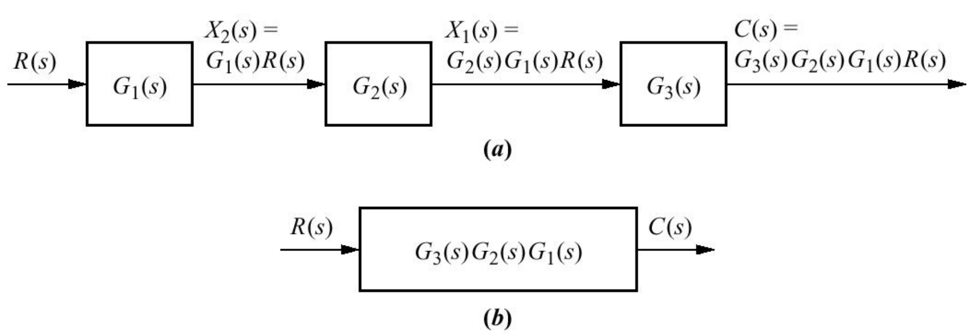


Figure Cascade (Series) Connection.

Parallel Connections

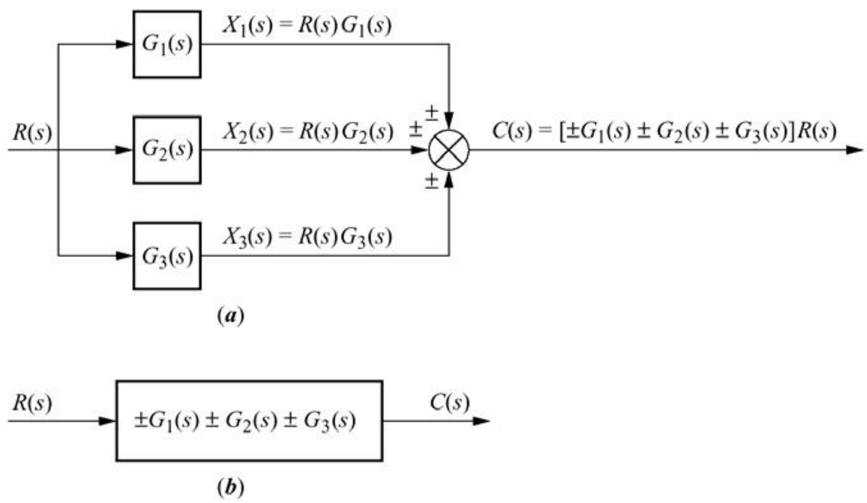


Figure Parallel Connection.

Closed Loop Transfer Function (Feedback Connections)

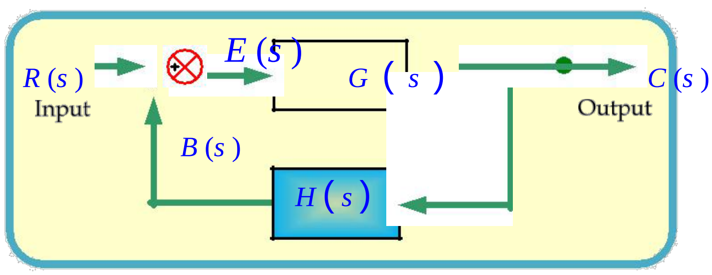


Figure (Repeated) Feedback connection

For the system shown in Figure 10-4, the output $C(s)$ and input $R(s)$ are related as follows:

$$C(s) = G(s)E(s)$$

where

$$E(s) = R(s) - B(s) = R(s) - H(s)C(s)$$

Eliminating $E(s)$ from these equations gives

$$C(s) = G(s)[R(s) - H(s)C(s)]$$

This can be written in the form

$$[1 + G(s)H(s)]C(s) = G(s)R(s)$$

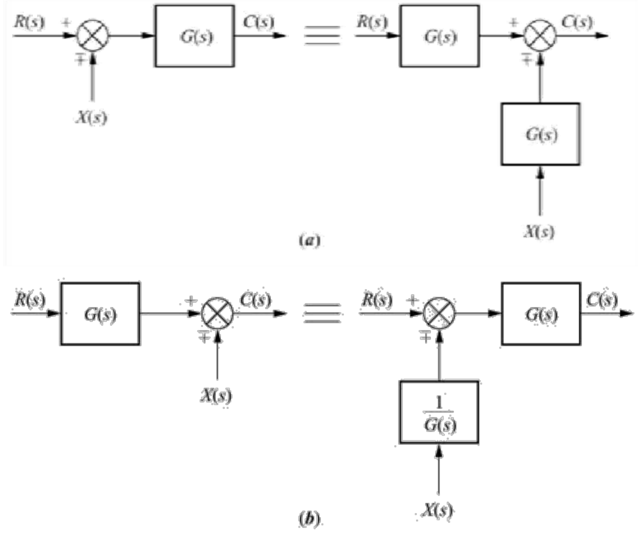
or

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

The ***Characteristic equation*** of the system is defined as an equation obtained by *setting the denominator polynomial of the transfer function to zero*. The ***Characteristic equation*** for the above system is

$$\mathbf{1 + G(s)H(s) = 0}.$$

Block Diagram Algebra for Summing Junctions



$C = G\,(+R \pm X)$ $= +GR \pm GX$	
$C = GR \pm X$ $= G\,(+R \pm X/G)$	

Figure Summing junctions.

Block Diagram Algebra for Branch Point

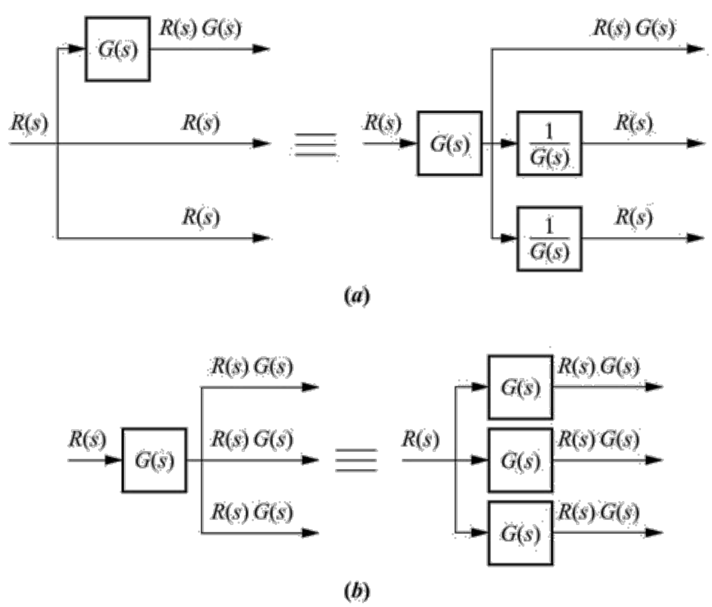


Figure Summing junctions.

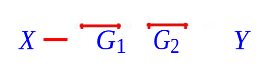
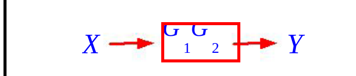
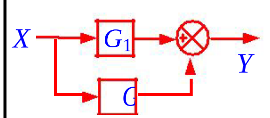
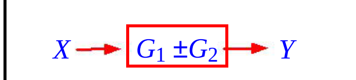
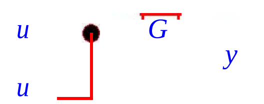
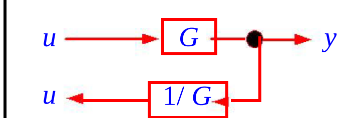
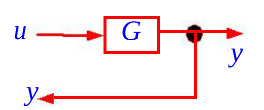
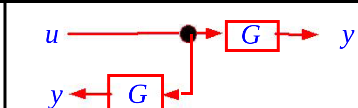
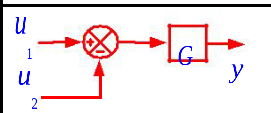
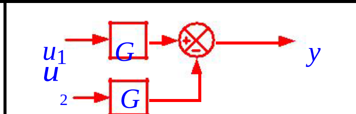
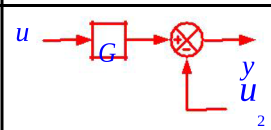
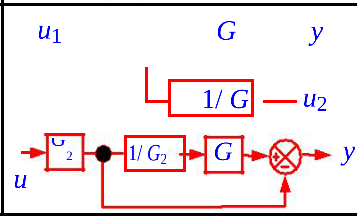
Block Diagram Reduction Rules

In many practical situations, the block diagram of a Single Input Single Output (SISO), feedback control system may involve several feedback loops and summing points. In principle, the block diagram of (SISO) closed loop system, no matter how complicated it is, it can be reduced to the standard single loop form shown in Figure 104. The basic approach to simplify a block diagram can be summarized in Table 1:

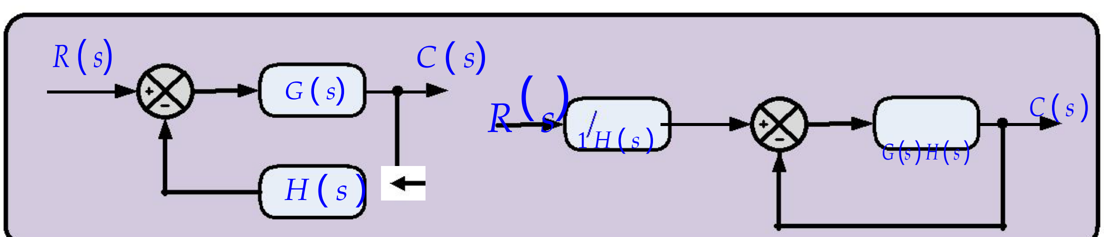
TABLE 10-1 Block Diagram Reduction Rules

1.	Combine all cascade blocks
2.	Combine all parallel blocks
3.	Eliminate all minor (interior) feedback loops
4.	Shift summing points to left
5.	Shift takeoff points to the right
6.	Repeat Steps 1 to 5 until the canonical form is obtained

TABLE 10-2. Some Basic Rules with Block Diagram Transformation

	Manipulation	Original Block Diagram	Equivalent Block Diagram	Equation
1	Combining Blocks in Cascade			$Y = (G_1 G_2)X$
2	Combining Blocks in Parallel; or Eliminating a Forward Loop			$Y = (G_1 \pm G_2)X$
3	Moving a pickoff point behind a block			$y = G u$ $u = \frac{1}{G} y$
4	Moving a pickoff point ahead of a block			$y = Gu$
5	Moving a summing point behind a block			$e_2 = G(u_1 - u_2)$
6	Moving a summing point ahead of a block			$y = Gu_1 - u_2$ $y = (G_1 - G_2)u$

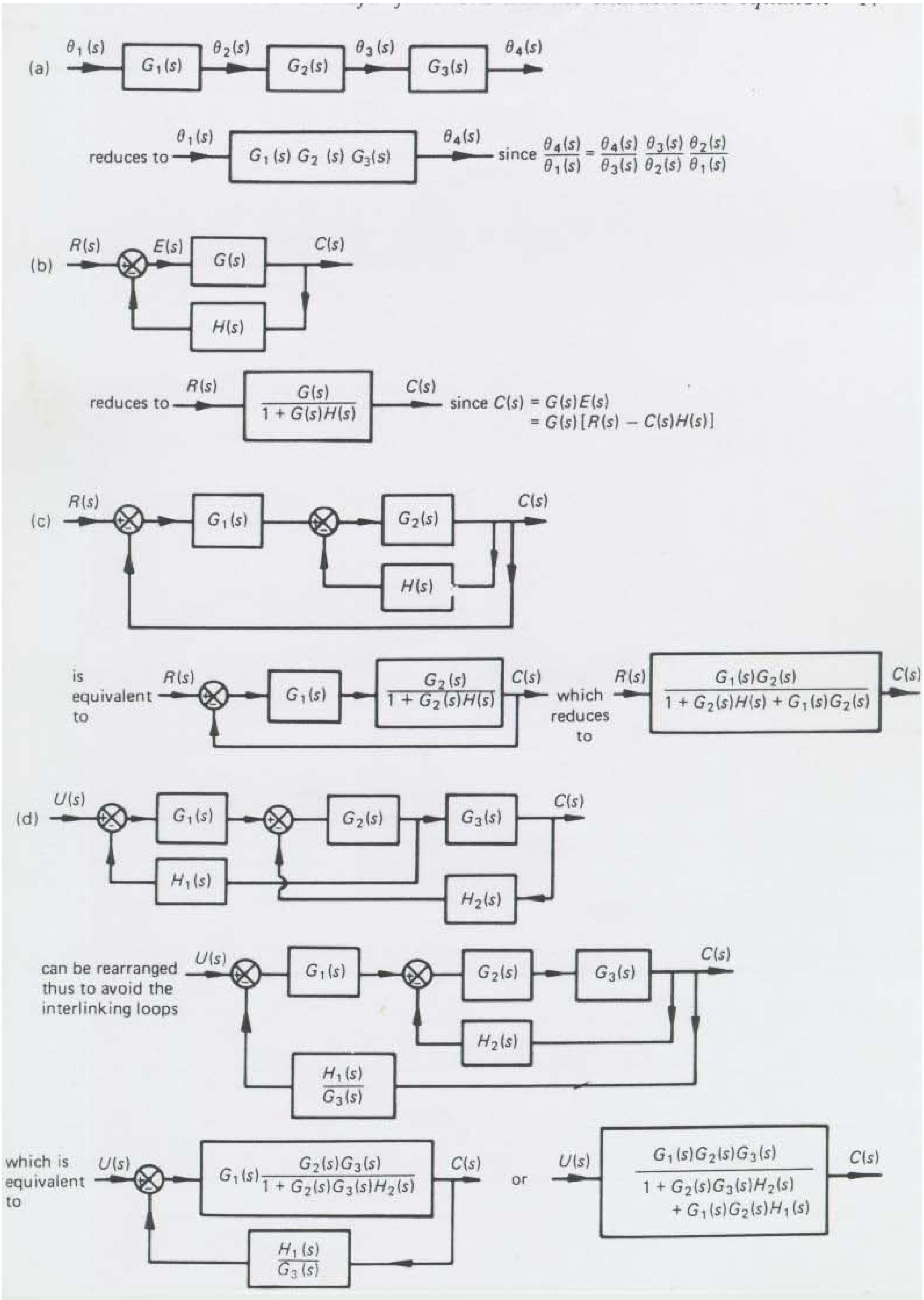
Example 1: A feedback system is transformed into a unity feedback system



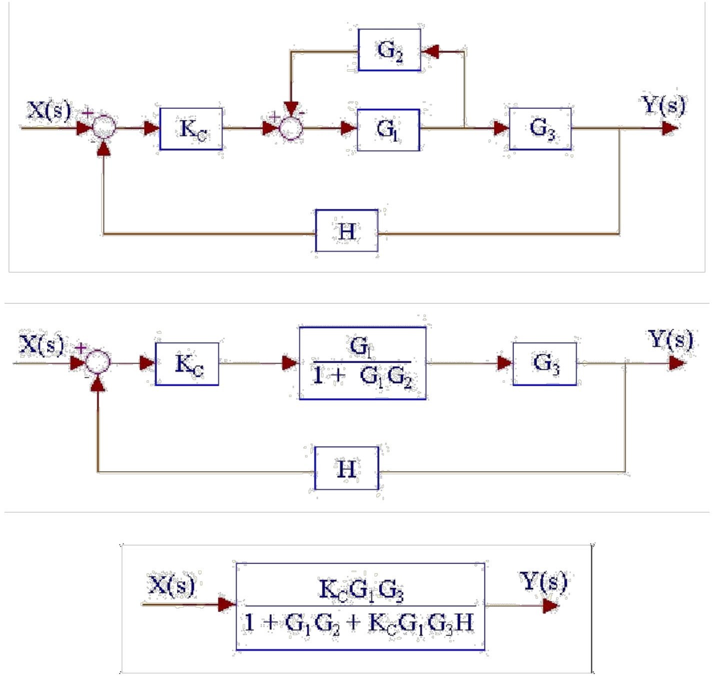
$$\frac{C}{R} = \frac{G}{1 \pm GH} = \frac{1}{H} \cdot \frac{GH}{1 \pm GH}$$ =Closed-loop Transfer function

Example 2:

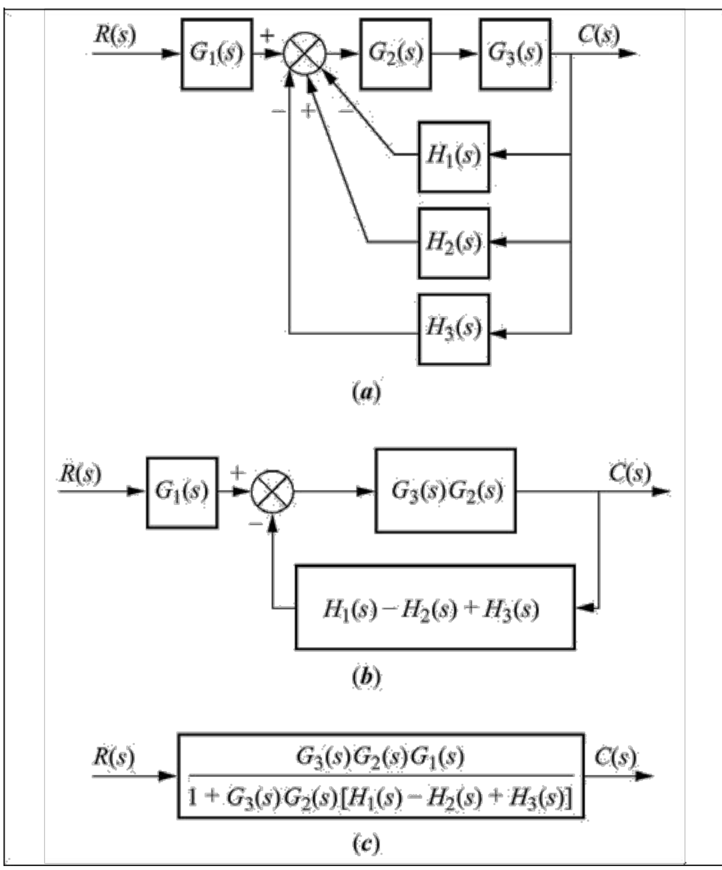
Reduce the following block diagrams



Example 3:



Example 4



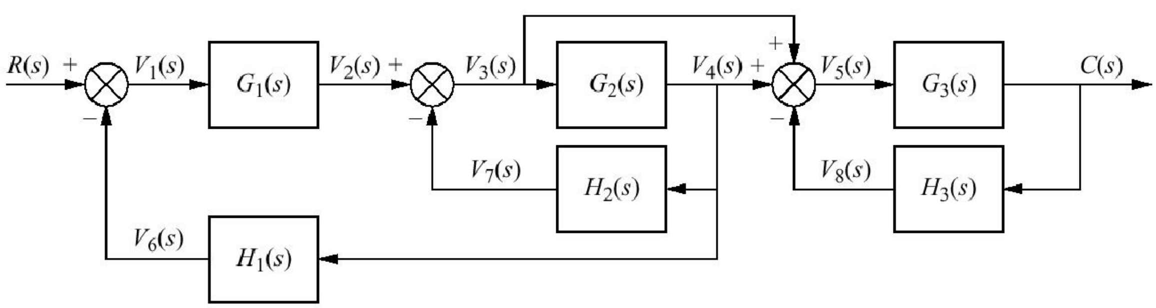
G_1 and G_2 are in series

H_1 and H_2 and H_3 are in parallel

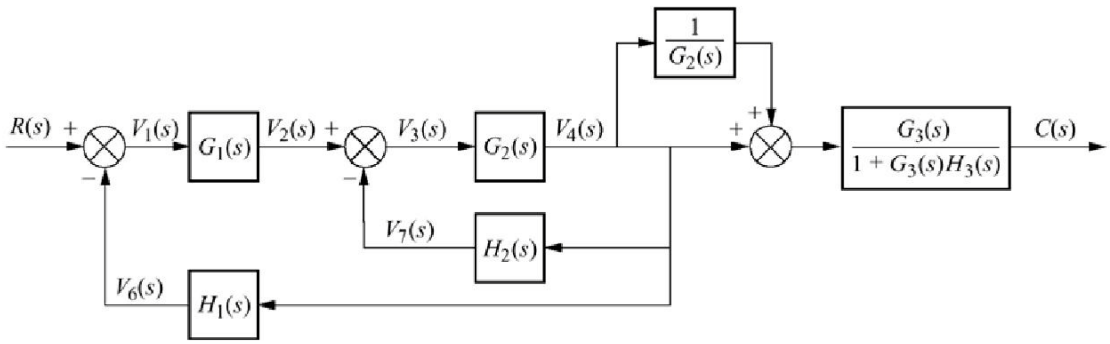
G_1 is in series with the feedback configuration.

$$\frac{C(s)}{R(s)} = G_1 \frac{G_3 G_2}{1 + G_3 G_2 (H_1 - H_2 + H_3)}$$

Example 5: The main problem here is the feedforward of $V_3(s)$. Solution is to move this pickoff point forward.

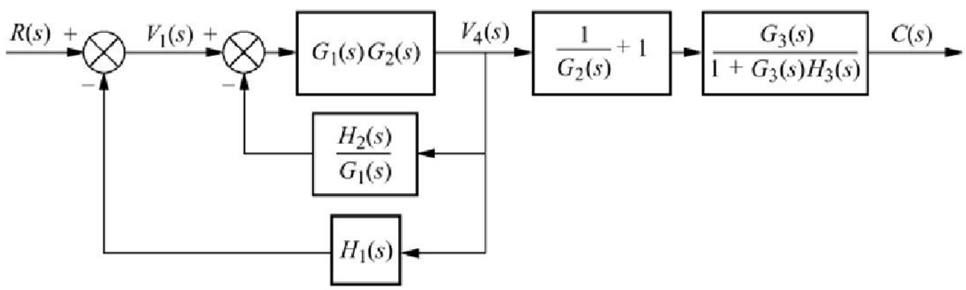


$R(s)$
$V_1(s)$
$G_1(s)$
$V_2(s)$
$V_3(s)$
$G_2(s)$
$V_4(s)$
$V_5(s)$
$G_3(s)$
$C(s)$
$V_7(s)$
$H_2(s)$
$V_8(s)$
$H_3(s)$
$V_6(s)$
$H_1(s)$



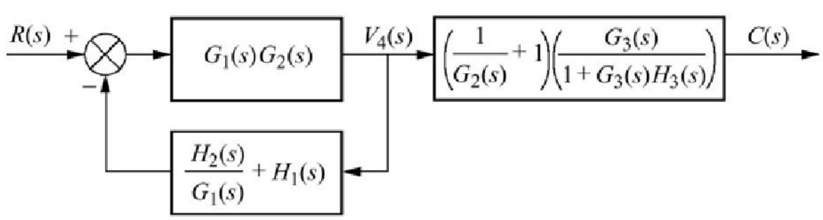
$\frac{1}{G_2(s)}$
$R(s)$
$V_1(s)$
$G_1(s)$
$V_2(s)$
$V_3(s)$
$G_2(s)$
$V_4(s)$
$\frac{G_3(s)}{1+G_3(s)H_3(s)}$
$C(s)$
$V_7(s)$
$H_2(s)$
$V_6(s)$
$H_1(s)$

(*a*)



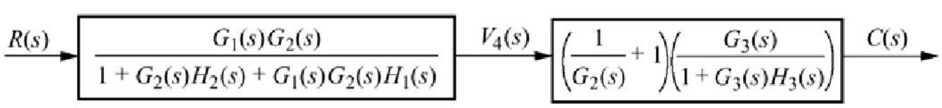
$R(s)$
$V_1(s)$
$G_1(s)G_2(s)$
$V_4(s)$
$\frac{1}{G_2(s)}+1$
$\frac{G_3(s)}{1+G_3(s)H_3(s)}$
$C(s)$
$\frac{H_2(s)}{G_1(s)}$
$H_1(s)$

(*b*)



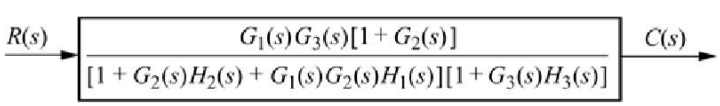
$R(s)$
$G_1(s)G_2(s)$
$V_4(s)$
$\left(\frac{1}{G_2(s)}+1\right)\left(\frac{G_3(s)}{1+G_3(s)H_3(s)}\right)$
$C(s)$
$\frac{H_2(s)}{G_1(s)}+H_1(s)$

(*c*)



$R(s)$
$\frac{G_1(s)G_2(s)}{1+G_2(s)H_2(s)+G_1(s)G_2(s)H_1(s)}$
$V_4(s)$
$\left(\frac{1}{G_2(s)}+1\right)\left(\frac{G_3(s)}{1+G_3(s)H_3(s)}\right)$
$C(s)$

(*d*)



$R(s)$
$\frac{G_1(s)G_3(s)[1+G_2(s)]}{[1+G_2(s)H_2(s)+G_1(s)G_2(s)H_1(s)][1+G_3(s)H_3(s)]}$
$C(s)$

(*e*)

Example 6:

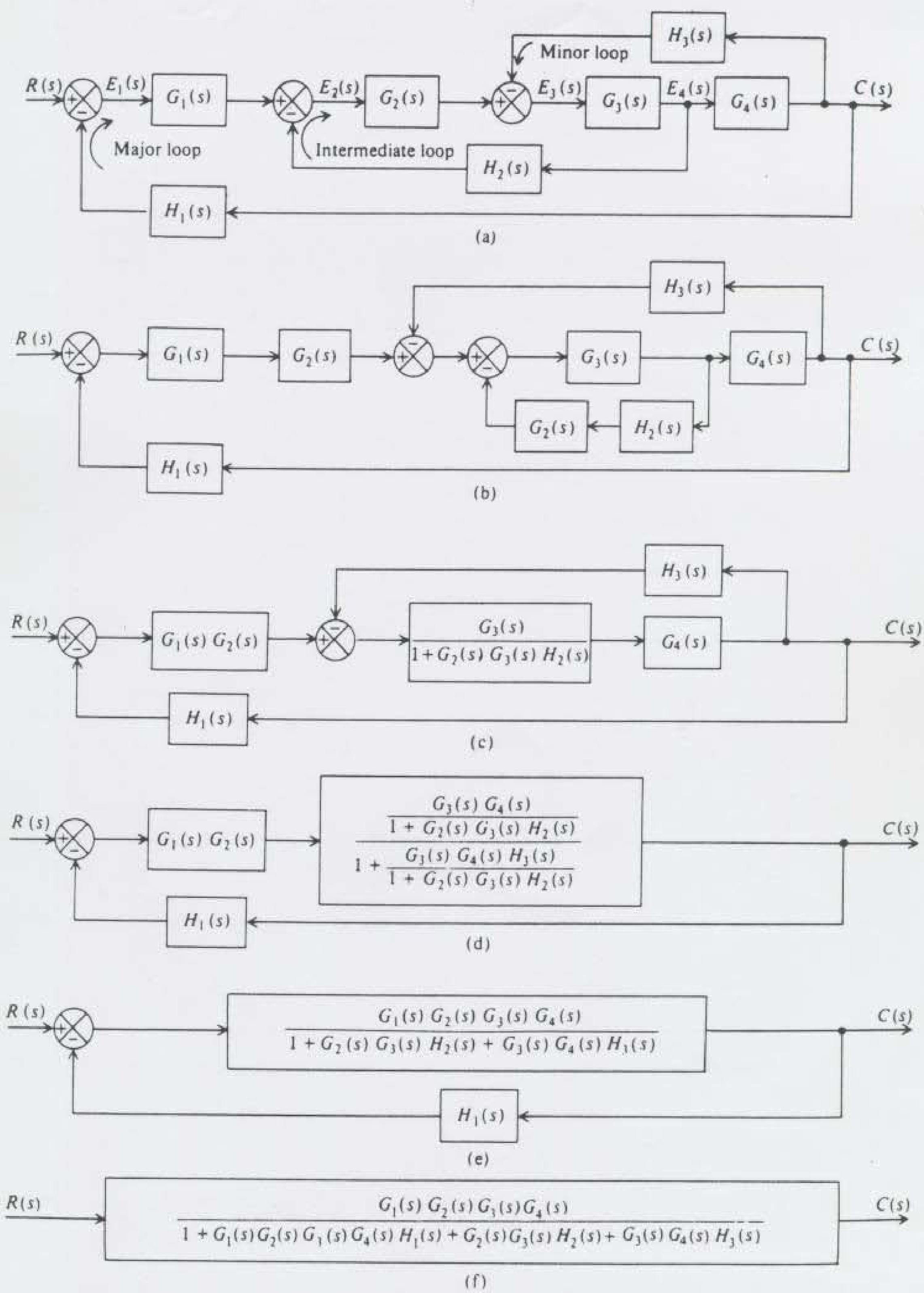


Fig. 2.13 Reducing a multiple-loop system containing complex paths. (a) The original system. (b) Rearrangement of the summing points of the intermediate and minor loops. (c) Reduction of the equivalent intermediate loop. (d) Reduction of the equivalent minor loop. (e) The equivalent feedback system. (f) The system transfer function.

BLOCK DIAGRAM

1.18. BLOCK DIAGRAM REPRESENTATION

Any system can be described by a set of differential equations or can be represented by the schematic diagram containing all components and their connections. But for the complicated systems these two methods are not suitable. The block diagram representation is the combination of above two methods. A block may represent a single component or a group of components, but each block is completely characterised by a transfer function. The transfer function is an expression which relates output to input in s-domain. Transfer function does not give any information about the internal structure of the system. Once we determine the transfer function, then we can represent the system by the block diagram. Block diagrams are single line diagram, that is the flow of system variables from one bolck to another block is represented by a single line. Figure 1.66 shows the block diagram representation of a system.

where,

$R(s)$ = input

$C(s)$ = output

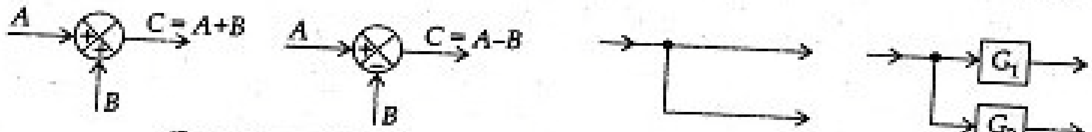
$G(s)$ = transfer function

Fig. 1.66.

Then, the system can be represented as

$$C(s) = R(s).\ G(s) \qquad ...(1.101)$$

The flow of system variables from one block to another block is represented by the arrow. In addition to this, the sum of the signals or the difference of the signals are represented by a summing point (Fig. 1.67(*a*). Application of one input source to two or more blocks is represented by a take off point (Fig. 1.67(*b*).



(Summing point) (Take off point)

Fig. 1.67.

1.19. HOW TO DRAW THE BLOCK DIAGRAM

Consider a simple R-L circuit shown in Fig. 1.68

Apply KVL

$$V_i = Ri + L\frac{di}{dt} \qquad ...(1.102)$$

$$V_0 = L\frac{di}{dt} \qquad ...(1.103)$$

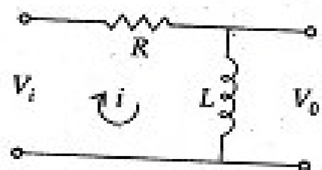


Fig. 1.68.

Laplace transform of equation (1.102) and (1.103) with initial condition zero

$$V_i(s) = I(s)\,R + SL\,I(s)$$

$$V_i(s) = I(s)\,(R + SL) \qquad ...(1.104)$$

$$V_0(s) = SL\,I(s) \qquad ...(1.105)$$

From (1.104) and (1.105)

$$\frac{V_0(s)}{V_i(s)} = \frac{sL}{R+sL} \qquad ...(1.106)$$

Form Fig. 1.68 $$i = \frac{V_i - V_0}{R} \quad ...(1.107)$$

$$V_0 = L\frac{di}{dt} \quad ...(1.108)$$

laplace transform of equations (1.107) and (1.108)

$$I(s) = \frac{1}{R}\,[V_i(s) - V_0(s)] \quad ...(1.109)$$

$$V_0(s) = SL\,I(s) \quad ...(1.110)$$

For right hand side of equation (1.109) we use a summing point.

The output of the summing point is given to block and output of the block is $I(s)$ as per equation (1.109).

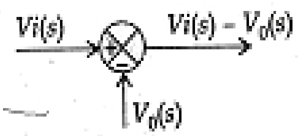


Fig. 1.69.

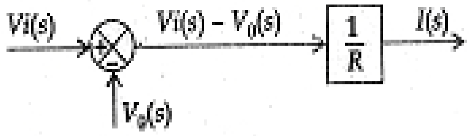


Fig. 1.70.

Form equation (1.110) the output of block $I(s)$ is given to another block containing the element SL and the output of the second block is V_0.

Combining the Fig. 1.70 and 1.71 we get required block diag.

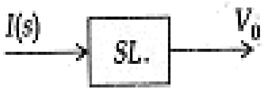


Fig. 1.71.

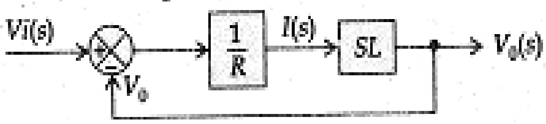


Fig. 1.72.

EXAMPLE 1.27. Draw the block diagram of series RLC circuit, where V_i and V_0 are the input and output voltages.

Solution : The transformed network in s-domain is shown in Fig. 1.74.

Fig. 1.73

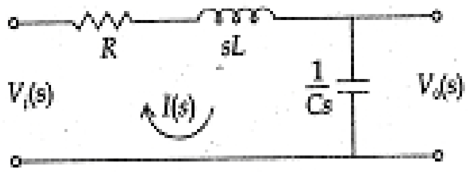


Fig. 1.74.

From Fig. 1.74 : $$I(s) = \frac{V_i(s) - V_0(s)}{R + sL} \quad ...(1.111)$$

$$V_0(s) = \frac{1}{Cs}\,I(s) \quad ...(1.112)$$

For *R.H.S.* of equation (1.111), we require a summing point

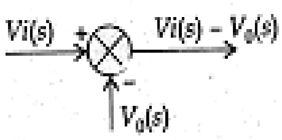


Fig. 1.75.

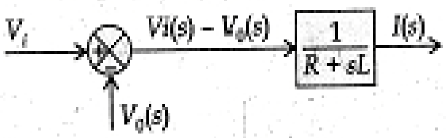


Fig. 1.76.

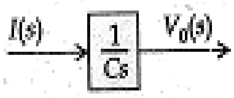


Fig. 1.77.

Combining the Figs. 1.76 and 1.77 we get required block diag.

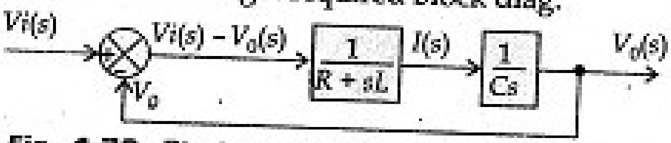


Fig. 1.78. Block diagram of series RLC circuit

EXAMPLE 1.28. Draw the block diagram of the circuit shown in Fig. 1.79.

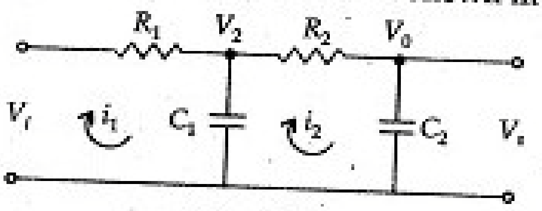


Fig. 1.79.

Solution :

$$i_1(t) = \frac{V_i(t) - V_2(t)}{R_1} \quad \text{...(1.113a)}$$

$$V_2(t) = \frac{1}{C_1}\int[i_1(t) - i_2(t)]dt \quad \text{...(1.113b)}$$

$$i_2(t) = \frac{1}{R_2}\,[V_2(t) - V_0(t)] \quad \text{...(1.113c)}$$

$$V_0(t) = \frac{1}{C_2}\int i_2\,dt \quad \text{...(1.113d)}$$

Take the Laplace transform of above equation

$$I_1(s) = \frac{1}{R_1}[V_i(s) - V_2(s)] \quad \text{...(1.113e)}$$

$$V_2(s) = \frac{1}{sC_1}[\,I_1(s) - I_2(s)] \quad \text{...(1.113f)}$$

$$I_2(s) = \frac{1}{R_2}\,[V_2(s) - V_0(s)] \quad \text{...(1.113g)}$$

$$V_0(s) = \frac{1}{sC_2}I_2(s) \quad \text{...(1.113h)}$$

From equation (5)

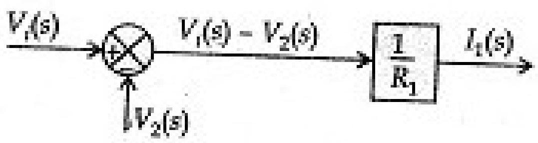


Fig. 1.80 *(a)*

From equation (6)

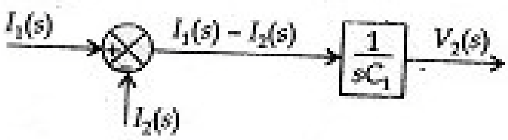


Fig. 1.80 *(b)*

From equation (7)

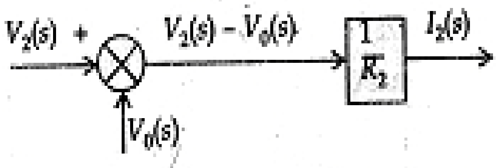


Fig. 1.80 (c)

From equation (8)

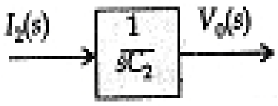


Fig. 1.80 (d)

combining all the Fig. 1.80(*a*), 1.80 (*b*), 1.80 (*c*) and 1.80 (*d*)

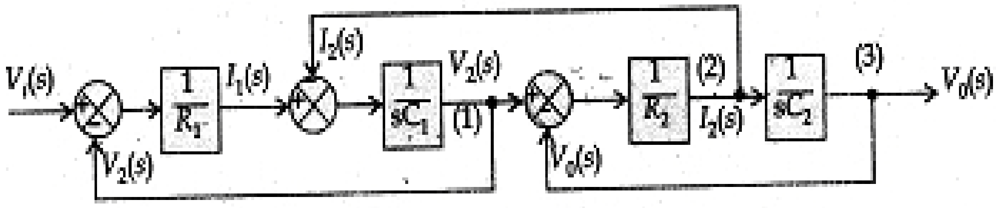


Fig. 1.80 (e)

EXAMPLE 1.29. Draw the block diag. for the circuit shown in Fig. 1.81, where V_i and i_L are the input and output variables respectively.

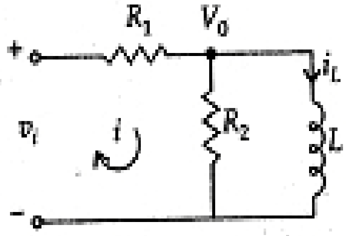


Fig. 1.81.

Solution : From Fig. 1.81

$$i = \frac{V_i - V_0}{R_1} \quad \text{...(1.114a)}$$

$$i - i_L = \frac{V_0}{R_2} \quad \text{...(1.114b)}$$

$$V_0 = L\frac{di_L}{dt} \quad \text{...(1.114c)}$$

Laplace transform of above equation

$$I(s) = \frac{1}{R_1}[V_i(s) - V_0(s)] \quad \text{...(1.114d)}$$

$$V_0(s) = R_2[I(s) - I_L(s)] \quad \text{...(1.114e)}$$

$$V_0(s) = SLI_L(s)$$

$$I_L(s) = \frac{1}{SL}V_0(s) \quad \text{...(1.114f)}$$

From 1.114 (*d*), 1.114 (*e*) and (1.114), *f*, we can draw the block diagram.

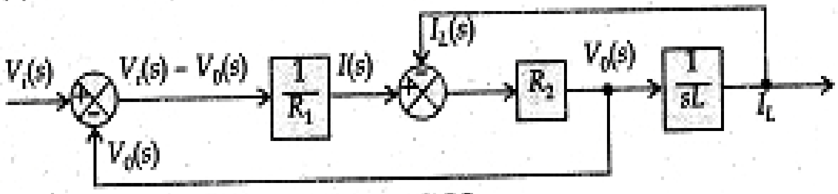


Fig. 1.82.

1.20. CLOSED LOOP CONTROL SYSTEM

A closed loop system is one in which output is fed back into an error detector and compared with the reference input. The feedback may be negative or positive.

Consider a closed loop system shown in Fig. 1.83

where, $R(s)$ = Reference input

$E(s)$ = Actuating signal or error signal

$G(s)$ = Forward path transfer function

$C(s)$ = Output signal

$H(s)$ = Feedback transfer function

$B(s)$ = Feedback signal

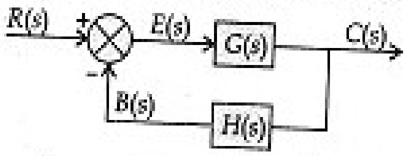


Fig. 1.83.

From Fig. 1.83

$$C(s) = G(s).E(s) \quad ...(1.115a)$$

$$B(s) = H(s).C(s) \quad ...(1.115b)$$

$$E(s) = R(s) - B(s) \quad ...(1.115c)$$

Put the value of $C(s)$ from equation (1.115(a)) in equation (1.115(b))

$$B(s) = H(s) \cdot G(s)\, E(s)$$

$$\frac{B(s)}{E(s)} = G(s) \cdot H(s)$$

$$\frac{B(s)}{E(s)} = \text{open loop transfer function} = G(s)\, H(s) \quad ...(1.115d)$$

Put the value of $E(s)$ from equations 1.115 (c) in 1.115 (a)

$$C(s) = G(s)\,[R(s) - B(s)]$$

$$C(s) = R(s).\,G(s) - G(s).\,B(s) \quad ...(1.115e)$$

Put the value of $B(s)$ from 1.115b in 1.115e

$$C(s) = R(s)\,G(s) - G(s)\,H(s).\,C(s)$$

or $$C(s)\,[1 + G(s)\,H(s)] = R(s)\,G(s)$$

or $$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)\,H(s)} \quad ...(1.115f)$$

$$\frac{C(s)}{R(s)} = M(s) = \text{closed loop transfer function} = \frac{G(s)}{1 + G(s)\,H(s)}$$

If the feedback is positive, then equation (1.115 (f)) becomes

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 - G(s)\,H(s)} \quad ...(1.115g)$$

From equation (1.115 (a)) put the value of $C(s)$ in equation (1.115(f))

$$\frac{G(s)\,E(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

or $$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)H(s)} \quad ...(1.115h)$$

$$\frac{E(s)}{R(s)} = \text{Error ratio} = \frac{1}{1 + G(s)H(s)}$$

For positive feedback

$$\frac{E(s)}{R(s)} = \frac{1}{1 - G(s)H(s)} \quad ...(1.115i)$$

Put the value of $C(s)$ from equation (1.115 (*a*)) in (1.115(*b*))

$$B(s) = H(s).\,G(s).\,E(s)$$

Put the value of $E(s)$ from (1.115 (*c*)) in above equation

$$B(s) = H(s).\,G(s)\,[R(s) - B(s)] \quad \text{...(1.115}j\text{)}$$

from equation (1.115*j*)

$$\frac{B(s)}{R(s)} = \frac{G(s)\,H(s)}{1+G(s)\,H(s)} \quad \text{...(1.115}k\text{)}$$

$\frac{B(s)}{R(s)}$ = Primary feedback ratio = $\frac{G(s)\,H(s)}{1+G(s)\,H(s)}$

For positive feedback

$$\frac{B(s)}{R(s)} = \frac{G(s)\,H(s)}{1-G(s)\,H(s)} \quad \text{...(1.115}l\text{)}$$

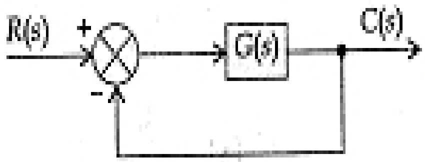


Fig. 1.84.

A unity feedback control system is shown in Fig. 1.84.

For unity feedback control system $H(s) = 1$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} \quad \text{For negative feedback}$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1-G(s)} \quad \text{For positive feedback}$$

1.24. SIGNAL FLOW GRAPH

The process of block diagram reduction technique is time consuming because at every stage modified block diagram is to be redrawn.

A simple method was developed by S.J. Mason which is known as signal flow graph. This method is very simple and does not require any reduction technique. Signal flow graph is applicable to the linear systems.

A signal flow graph is a diagram which represents a set of simultaneous equations.

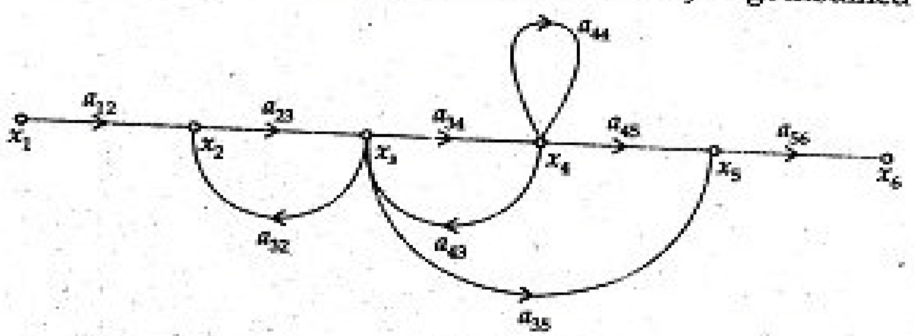


Fig. 1.101

Signal flow graph consists of nodes and these nodes are connected by a directed line called ranches. Every branch of signal flow graph having an arrow, which reprsents the flow of signal.

The following terms are associated with the signal flow graph.

1. **Input node or source node :** An input node is a node which has only outgoing branches. For example x_1 is the input node.
2. **Output node or sink node :** An output node is a node that has only one or more incoming branches.*e.g.* x_6 is the output node.
3. **Mixed nodes :** A node having incoming and outgoing branches is known as mixed nodes. For example x_2, x_3, x_4, and x_5 are the mixed nodes.
4. **Transmittance :** Transmittance also known as transfer function, which is normally written on the branch near the arrow. For example a_{12}, a_{23} etc.
5. **Forward path :** Forward path is a path which originates from the input node and terminates at the output node and along which no node is traversed more than once.

   For example in Fig 1.101 there are two forward paths.

   1. x_1 to x_2 to x_3 to x_4 to x_5 to x_6
   2. x_1 to x_2 to x_3 to x_5 to x_6
6. **Loop :** Loop is a path that originates and terminates on the same node and along which no other node is traversed more than once.

   For example x_2 to x_3 to x_2
   x_3 to x_4 to x_3
7. **Self loop :** It is a path which originates and terminates on the same node. For example x_4 to x_4
8. **Path gain :** The product of the branch gains along the path is called path gain. For example the gain of the path x_1 to x_2 to x_3 to x_4 to x_5 to x_6 is $a_{12}\, a_{23}\, a_{34}\, a_{45}\, a_{56}$
9. **Loop gain :** The gain of the loop is known as loop gain. For example the gain of the loop x_2 to x_3 to x_2 is $a_{23}\, a_{32}$.
10. **Non-touching loops :** Non touching loops having no common nodes branch and paths. For example the loops x_2 to x_3 to x_2, and x_4 to x_4 are non-touching loops.

1.25. PROPERTIES OF SIGNAL FLOW GRAPH

1. Signal flow graph is applicable to linear time-invariant systems.
2. The signal flow is only along the direction of arrows.
3. The value of variable at each node is equal to the algebraic sum of all signals entering at that node.
4. The gain of signal flow graph is given by Mason's formula.
5. The signal gets multiplied by the branch gain when it travels along it.
6. The signal flow graph is not be the unique property of the system.

1.26. COMPARISON OF BLOCK DIAGRAM AND SIGNAL FLOW GRAPH METHOD

S.No.	Block Diagram	SFG
1.	Applicable to linear time invariant systems only.	Applicable to linear time invariant system.
2.	Each element is represented by block.	Each variable is represented by node.
3.	Summing point and take off points are separate.	Summing and take off points are absent.
4.	Self loop do not exist.	Self loop can be exist.
5.	It is time consuming method.	Require less time by using Mason gain formula.
6.	Block diagram is required at each & every step.	At each step it is not necessary to draw SFG.
7.	Transfer function of the element is shown inside the corresponding block.	Transfer function is shown along the branches connecting the nodes.
8.	Feedback path is present.	Feedback loops are used.

1.27. CONSTRUCTION OF SIGNAL FLOW GRAPH FROM EQUATIONS

Consider the following sets of equations

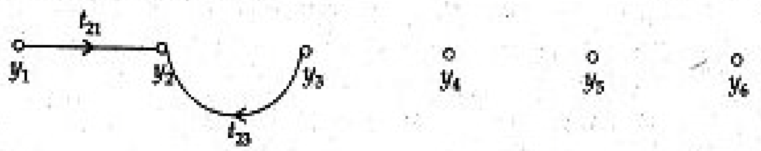
$$y_2 = t_{21} y_1 + t_{23} y_3$$
$$y_3 = t_{32} y_2 + t_{33} y_3 + t_{31} y_1$$
$$y_4 = t_{43} y_3 + t_{42} y_2$$
$$y_5 = t_{54} y_4$$
$$y_6 = t_{65} y_5 + t_{64} y_4$$

where y_1 is the input and y_6 is the output.

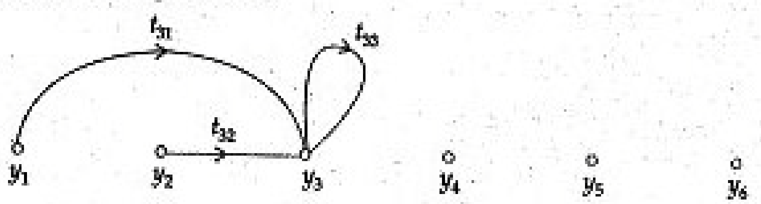
First of all draw the nodes. In the given example there are six nodes. From the first equation it is clear that the y_2 is the sum of two signals. Similarly, y_3 is the sum of three signals and so on. Insert the branches with proper transmittance to connect the nodes.

Step 1 : Draw the nodes

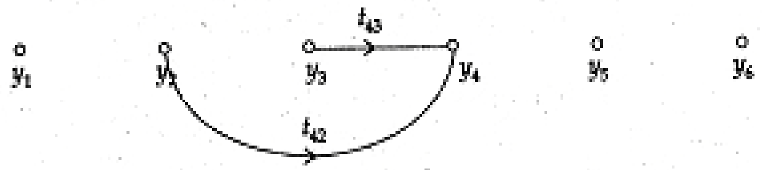
Step 2 : Draw the SFG for equation (1)



Step 3 : Draw the SFG for equation (2)



Step 4 : Draw SFG for equation (3)



Step 5 : Draw SFG for equation (4)

Step 6 : Draw SFG for equation (5)

Step 7 : Draw the complete signal flow graph with the help of above graphs.

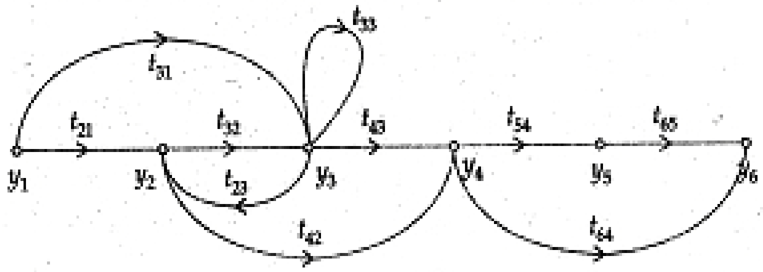


Fig. 1.102.

1.28. SIGNAL FLOW GRAPH FOR DIFFERENTIAL EQUATIONS

Consider the following defferential equation

$$y''' + 3y'' + 5y' + 2y = x \qquad ...(1.135)$$

Step 1 : Solve the eqn 1.135 for the highest order

$$y''' = x - 3y'' - 5y' - 2y$$

Step 2 : Conside. the left hand ..rm (highest order derivative) as dependent variable and all other terms on right hand side as independent variables.

construct the branches of signal flow graph as shown in Fig. (1.103(*a*).

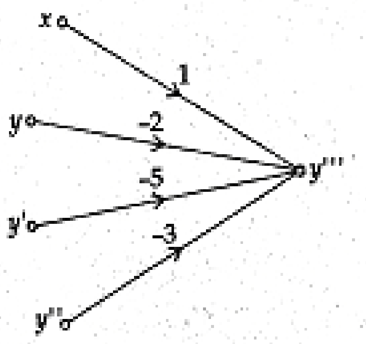


Fig. 1.103 (*a*)

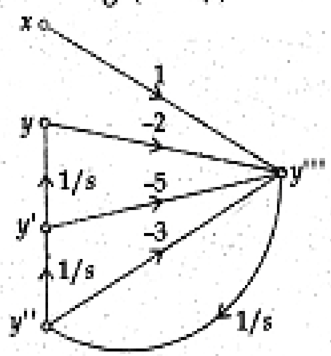
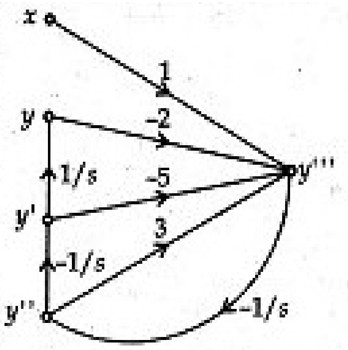


Fig. 1.103 (*b*)

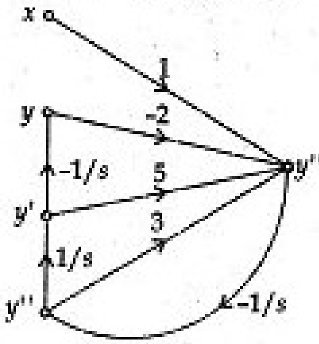


Step 3 : Connect the nodes of highest order derivative to the node whose order is lower than this and so on. The flow of the signal will be from higher node to the lower order node and transmittance will be $1/s$ as shown in Fig. 1.103 (*b*).

Step 4 : Reverse the sign of a branch connecting the p^{th} node to the q^{th} node of a signal flow graph without disturbing the transfer function.

Consider the Fig. 1.103(*b*), reverse the sign of the branch connecting y''' to y'', it is necessary to reverse the sign of all remaining branches entering as well as leaving the q^{th} node.

Similarly, reverse the sign of branch connecting y'' to y'.



By reversing the sign, we have already reverse the sign of branch connecting y' to y and therefore further reversal of sign is not required.

Step 5 : Redraw the signal flow graph (SFG).

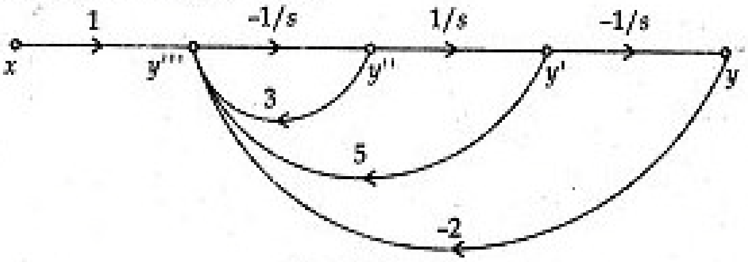


Fig. 1.103 (*c*)

1.29. CONSTRUCTION OF SIGNAL FLOW GRAPH FROM BLOCK DIAGRAM

Rules 1. All variables, summing points and take off points are represented by nodes.

2. If a summing point is placed before a take off point in the dirction of signal flow, in such case represent the summing point and takeoff point by a single node.

3. If a summing point is placed after a takeoff point in the direction of signal flow, in such case,

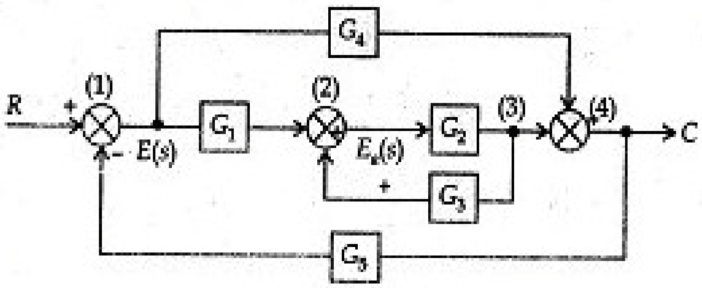
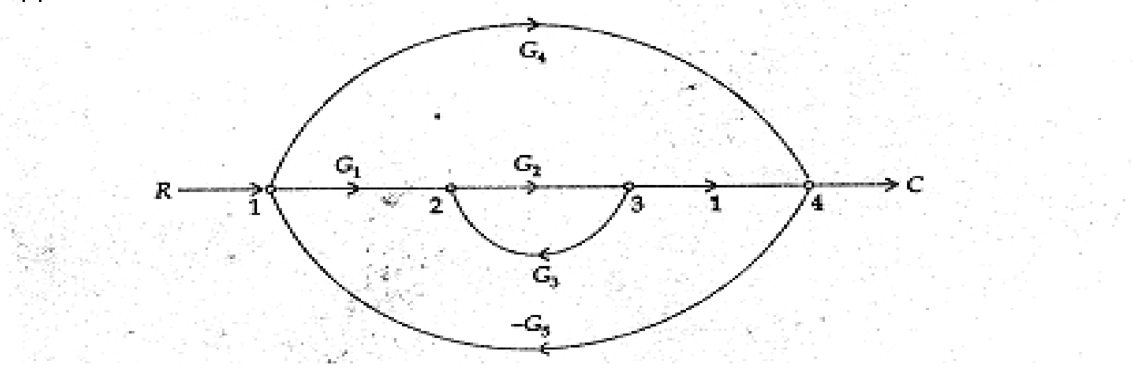


Fig. 1.104 (*a*)

represent the summing point and takeoff point by separate nodes connected by a branch having transmittance unity.

Consider the block diagram shown in Fig 1.104(*a*), the corresponding SFG is shown in Fig. 1.104 (*b*).



Example 1
Obtain the transfer function of C/R of the system whose signal flow graph is shown in Fig.1

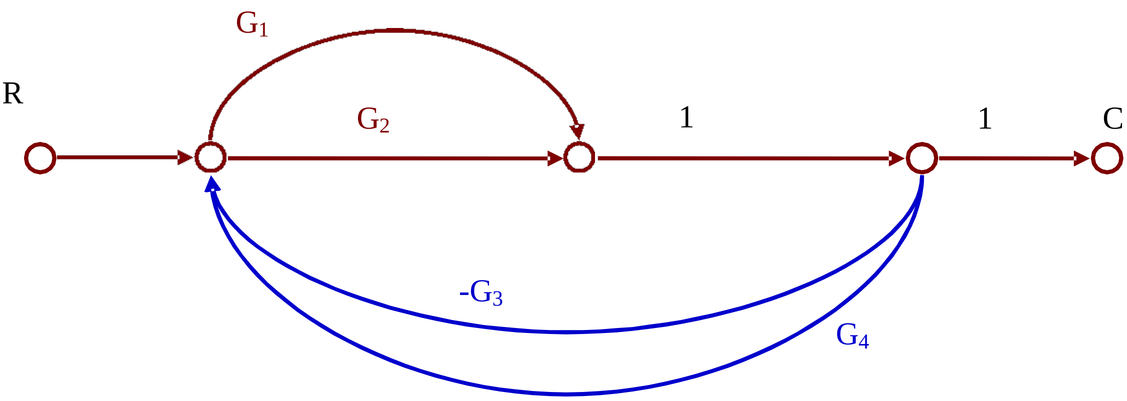


Figure 1 Signal flow graph of example 1

There are two forward paths: Gain of path 1 : $p_1=G_1$ Gain of path 2 : $P_2=G_2$

There are four loops with loop gains:
$L_1=-G_1G_3$, $L_2=G_1G_4$, $L_3= -G_2G_3$, $L_4= G_2G_4$ There are no non-touching loops.

$= 1+G_1G_3-G_1G_4+G_2G_3-G_2G_4$

Forward paths 1 and 2 touch all the loops. Therefore, $\Delta_1= 1$, $\Delta_2= 1$

The transfer function T = $\frac{C(s)}{R(s)} = \frac{P_1\Delta_1+P_2\Delta_2}{\Delta} = \frac{G_1+G_2}{1+G_1G_3-G_1G_4+G_2G_3-G_2G_4}$

Example 2

Obtain the transfer function of C(s)/R(s) of the system whose signal flow graph is shown in Fig.2.

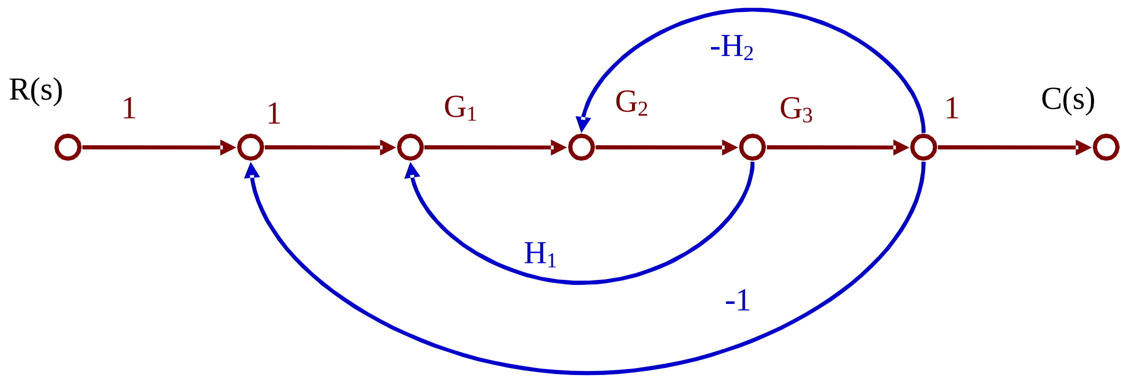


Figure 2 Signal flow graph of example 2

There is one forward path, whose gain is:
$P_1=G_1G_2G_3$ There are three loops with loop gains:
$L_1=-G_1G_2H_1$, $L_2=G_2G_3H_2$, $L_3=-G_1G_2G_3$ There are no non-touching loops.

$\Delta = 1 - G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3$

Forward path 1 touches all the loops. Therefore, $\Delta_1 = 1$.

The transfer function T = $\dfrac{C(s)}{R(s)} = \dfrac{P_1\Delta_1}{\Delta} = \dfrac{G_1G_2G_3}{1 - G_1G_2H_1 + G_1G_3H_2 + G_1G_2G_3}$

Example 3

Obtain the transfer function of C(s)/R(s) of the system whose signal flow graph is shown in Fig.3.

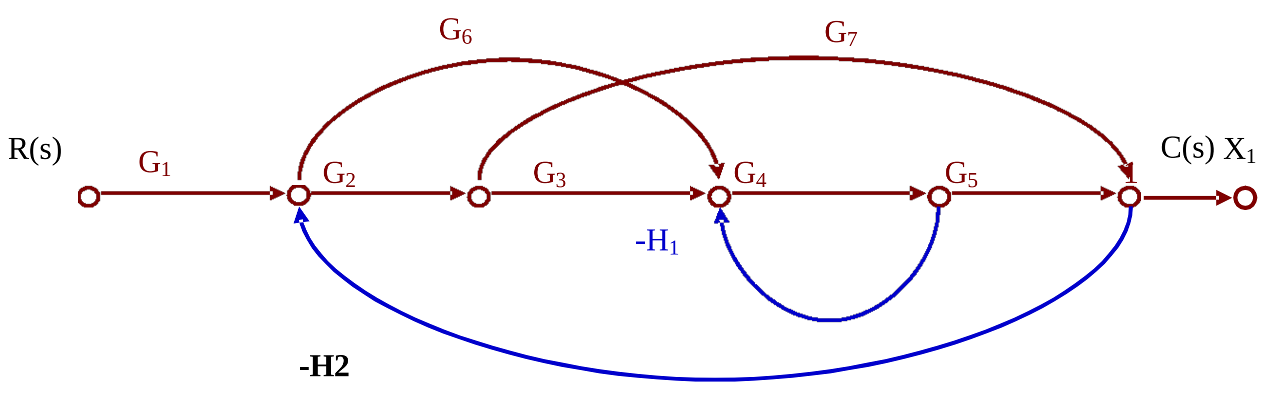


Figure 3 Signal flow graph of example 3

There are three forward paths.

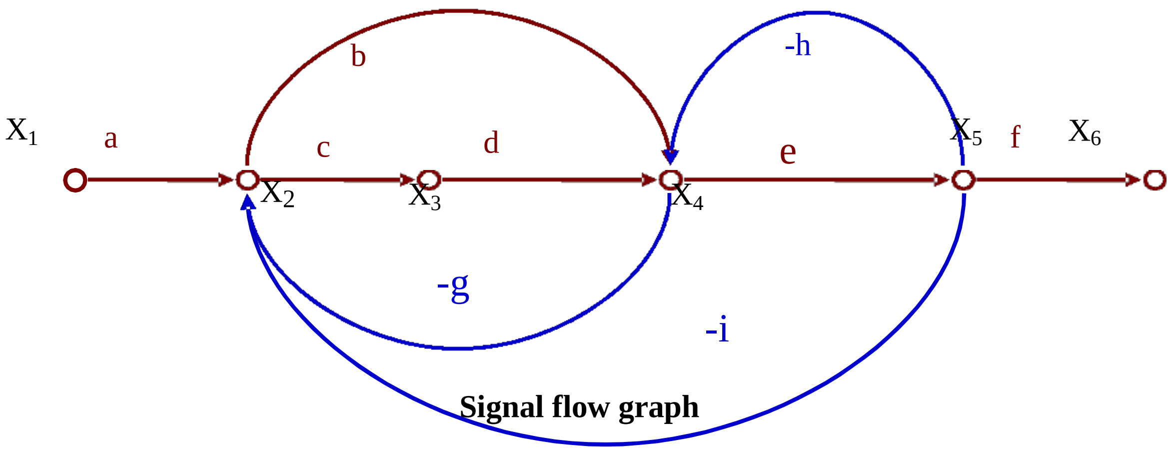
The gain of the forward path are:

$P_1 = G_1G_2G_3G_4G_5$ $P_2 = G_1G_6G_4G_5$
$P_3 = G_1G_2G_7$
There are four loops with loop gains:
$L_1 = -G_4H_1$, $L_2 = -G_2G_7H_2$, $L_3 = -G_6G_4G_5H_2$, $L_4 = -G_2G_3G_4G_5H_2$
There is one combination of Loops L_1 and L_2 which are nontouching with loop gain product $L_1L_2 = G_2G_7H_2G_4H_1$
$= 1+G_4H_1+G_2G_7H_2+G_6G_4G_5H_2+G_2G_3G_4G_5H_2+G_2G_7H_2G_4H_1$ Forward path 1 and 2 touch all the four loops. Therefore $\Delta_1 = 1$, $\Delta_2 = 1$. Forward path 3 is not in touch with loop1.
Hence, $\Delta_3 = 1+G_4H_1$.
The transfer function T =

$$\frac{C(s)}{R(s)} = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3}{\Delta} = \frac{G_1G_2G_3G_4G_5 + G_1G_4G_5G_6 + G_1G_2G_7(1+G_4H_1)}{1+G_4H_1+G_2G_7H_2+G_6G_4G_5H_2+G_2G_3G_4G_5H_2+G_2G_4G_7H_1H_2}$$

Example 4

Find the gains for the signal flow graph shown in Fig.4



Signal flow graph

There are two forward paths.
The gain of the forward path

$P_1 = acdef$ $P_2 = abef$
There are four loops with loop gains: $L_1 = -cdg$, $L_2 = -eh$,
$L_3 = -cdei$, $L_4 = -bei$
There is one combination of Loops L_1 and L_2 which are non touching with loop gain product $L_1L_2 = cdgeh$

= 1+cdg+eh+cdei+bei+cdgeh

Forward path 1 and 2 touch all the four loops. Therefore $\Delta_1 = 1$, $\Delta_2 = 1$.

The transfer function T = $\frac{X_6}{X_1} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{cdef + abef}{1 + cg + eh + cdei + bei + cdgeh}$

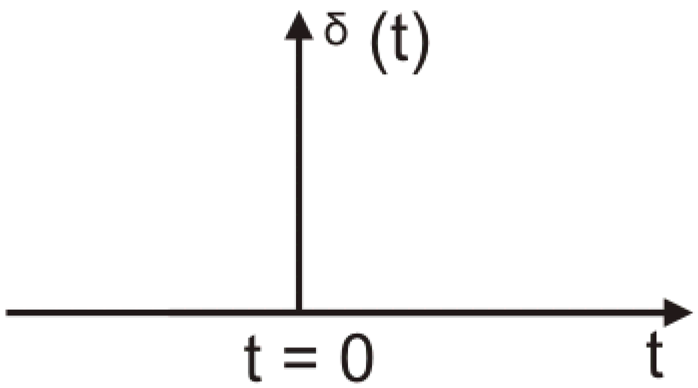
CHAPTER-5

Time Domain Analysis of Control Systems

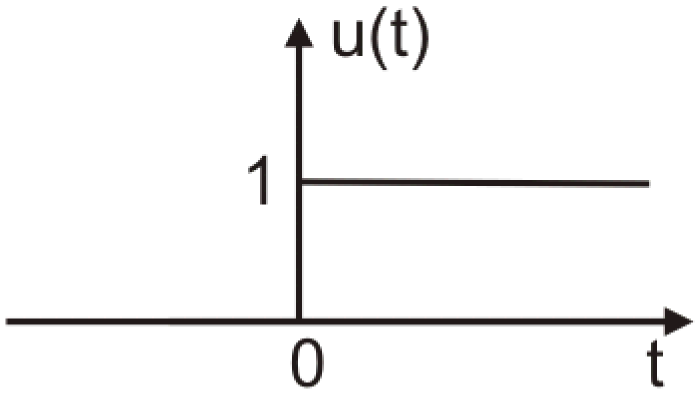
When we study the analysis of the **transient state and steady state response of control system** it is very essential to know a few basic terms and these are described below.

Standard Input Signals : These are also known as test input signals. The input signal is very complex in nature, it is complex because it may be a combination of various other signals. Thus it is very difficult to analyze characteristic performance of any system by applying these signals. So we use test signals or standard input signals which are very easy to deal with. We can easily analyze the characteristic performance of any system more easily as compared to non standard input signals. Now there are various types of standard input signals and they are written below:

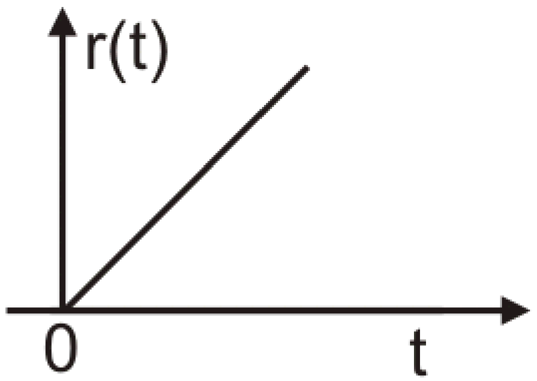
Unit Impulse Signal : In the time domain it is represented by ∂(t). The Laplace transformation of unit impulse function is 1 and the corresponding waveform associated with the unit impulse function is shown below.



Unit Step Signal : In the time domain it is represented by u (t). The Laplace transformation of unit step function is 1/s and the corresponding waveform associated with the unit step function is shown below.

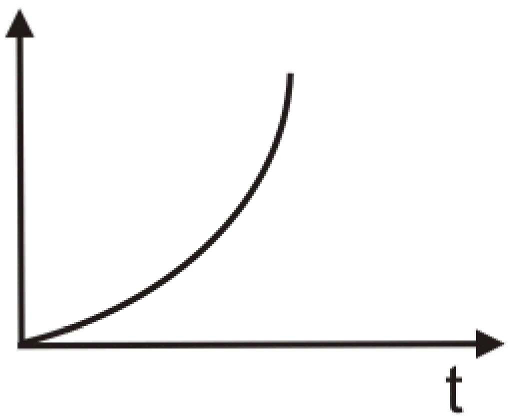


Unit Ramp signal : In the time domain it is represented by r (t). The Laplace transformation of unit ramp function is $1/s^2$ and the corresponding waveform associated with the unit ramp function is shown below.



Unit Ramp Signal

Parabolic Type Signal : In the time domain it is represented by $t^2 / 2$. The Laplace transformation of parabolic type of the function is $1 / s^3$ and the corresponding waveform associated with the parabolic type of the function is shown below.



Transient Response of Control System

As the name suggests **transient response of control system** means changing so, this occurs mainly after two conditions and these two conditions are written as follows-

- **Condition one :** Just after switching 'on' the system that means at the time of application of an input signal to the system.
- **Condition second :** Just after any abnormal conditions. Abnormal conditions may include sudden change in the load, short circuiting etc.

Steady State Response of Control System

Steady state occurs after the system becomes settled and at the steady system starts working normally. **Steady state response of control system** is a function of input signal and it is also called as forced response.

Now the transient state response of control system gives a clear description of how the system functions during **transient state and steady state response of control system**gives a clear description of how the system functions during steady state. Therefore the time analysis of both states is very essential. We will separately analyze both the types of responses. Let us first analyze the transient response. In order to analyze the transient response, we have some time specifications and they are written as follows:

Delay Time : This time is represented by t_d. The time required by the response to reach fifty percent of the final value for the first time, this time is known as delay time. Delay time is clearly shown in the time response specification curve.

Rise Time : This time is represented by t_r. We define rise time in two cases:

1. In case of under damped systems where the value of ζ is less than one, in this case rise time is defined as the time required by the response to reach from zero value to hundred percent value of final value.
2. In case of over damped systems where the value of ζ is greater than one, in this case rise time is defined as the time required by the response to reach from ten percent value to ninety percent value of final value.

Peak Time : This time is represented by t_p. The time required by the response to reach the peak value for the first time, this time is known as peak time. Peak time is clearly shown in the time response specification curve.

Settling Time : This time is represented by t_s. The time required by the response to reach and within the specified range of about (two percent to five percent) of its final value for the first time, this time is known as settling time. Settling time is clearly shown in the time response specification curve.

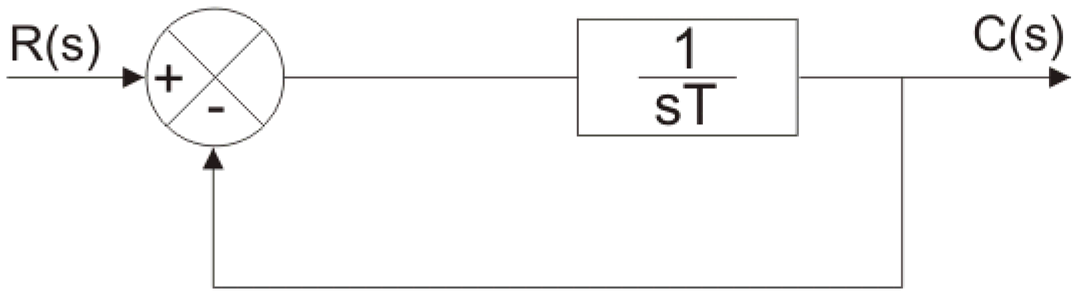
Maximum Overshoot : It is expressed (in general) in percentage of the steady state value and it is defined as the maximum positive deviation of the response from its desired value. Here desired value is steady state value.

Steady State Error : It can be defined as the difference between the actual output and the desired output as time tends to infinity.

Now we are in position we to do a time response analysis of a first order system.

Transient State and Steady State Response of First Order Control System

Let us consider the block diagram of the first order system.



From this block diagram we can find overall transfer function which is linear in nature. The transfer function of the first order system is 1/((sT+1)). We are going to analyze the steady state and transient response of control system for the following standard signal.

1. Unit impulse.
2. Unit step.
3. Unit ramp.

Unit impulse response : We have Laplace transform of the unit impulse is 1. Now let us give this standard input to a first order system, we have

$$C(s) = \frac{1}{1 + sT}$$

Now taking the inverse Laplace transform of the above equation, we have

$$c(t) = \frac{e^{-t/T}}{T}$$

It is clear that the **steady state response of control system** depends only on the time constant 'T' and it is decaying in nature.

Unit step response : We have Laplace transform of the unit impulse is 1/s. Now let us give this standard input to first order system, we have

$$C(s) = \frac{1}{s(1 + sT)}$$

With the help of partial fraction, taking the inverse Laplace transform of the above equation, we have

$$c(t) = 1 - e^{-t/T}$$

It is clear that the time response depends only on the time constant 'T'. In this case the steady state error is zero by putting the limit t is tending to zero.

Unit ramp response : We have Laplace transform of the unit impulse is $1/s^2$. Now let us give this standard input to first order system, we have

$$C(S) = \frac{1}{s^2(1 + sT)}$$

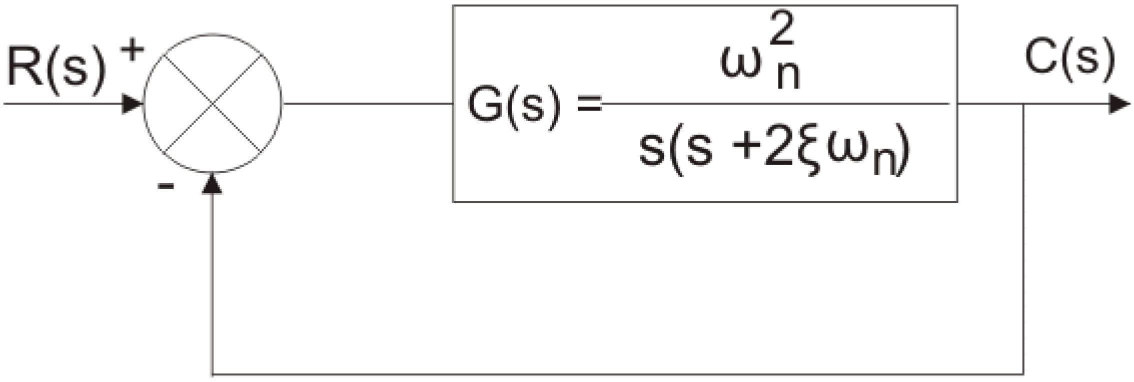
With the help of partial fraction, taking the inverse Laplace transform of the above equation we have

$$c(t) = 1 - T + Te^{-t/T}$$

On plotting the exponential function of time we have 'T' by putting the limit t is tending to zero.

Transient State and Steady State Response of Second Order Control System

Let us consider the block diagram of the second order system.



From this block diagram we can find overall transfer function which is nonlinear in nature. The transfer function of the second order system is $(\omega^2) / (s(s + 2\zeta\omega))$. We are going to analyze the **transient state response of control system** for the following standard signal.

Unit impulse response : We have Laplace transform of the unit impulse is 1. Now let us give this standard input to second order system, we have

$$C(S) = \frac{\omega^2}{s(s + 2\omega\zeta)}$$

Where ω is natural frequency in rad/sec and ζ is damping ratio.

Unit step response : We have Laplace transform of the unit impulse is 1/s. Now let us give this standard input to first order system, we have

$$C(S) = \frac{\omega^2}{s(s + 2\omega\zeta)}$$

With the help of partial fraction, taking the inverse Laplace transform of the above equation we have

$$c(t) = 1 - \frac{e^{-\zeta\omega t}\sin\left[\omega\sqrt{1-\zeta^2}\,t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right]}{\sqrt{1-\zeta^2}}$$

Now we will see the effect of different values of ζ on the response. We have three types of systems on the basis of different values of ζ.

1. **Under damped system :** A system is said to be under damped system when the value of ζ is less than one. In this case roots are complex in nature and the real parts are always negative. System is asymptotically stable. Rise time is lesser than the other system with the presence of finite overshoot.
2. **Critically damped system :** A system is said to be critically damped system when the value of ζ is one. In this case roots are real in nature and the real parts are always repetitive in nature. System is asymptotically stable. Rise time is less in this system and there is no presence of finite overshoot.
3. **Over damped system :** A system is said to be over damped system when the value of ζ is greater than one. In this case roots are real and distinct in nature and the real parts are always negative. System is asymptotically stable. Rise time is greater than the other system and there is no presence of finite overshoot.
4. **Sustained Oscillations :** A system is said to be sustain damped system when the value of zeta is zero. No damping occurs in this case.

Now let us derive the expressions for rise time, peak time, maximum overshoot, settling time and steady state error with a unit step input for second order system.

Rise time : In order to derive the expression for the rise time we have to equate the expression for c(t) = 1. From the above we have

$$c(t) = 1 = 1 - \frac{e^{-\zeta\omega t}\sin\left[\omega\sqrt{1-\zeta^2}t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right]}{\sqrt{1-\zeta^2}}$$

On solving above equation we have expression for rise time equal to

$$t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}}{\omega\sqrt{1-\zeta^2}}$$

Peak Time : On differentiating the expression of c(t) we can obtain the expression for peak time. dc(t)/ dt = 0 we have expression for peak time,

$$t_p = \frac{\pi}{\omega\sqrt{1-\zeta^2}}$$

Maximum overshoot : Now it is clear from the figure that the maximum overshoot will occur at peak time tp hence on putting the valye of peak time we will get maximum overshoot as

$$\%\,MP = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$$

Settling Time : Settling time is given by the expression

$$t_s = \frac{4}{\omega\zeta}$$

2.5. TRANSIENT RESPONSE SPECIFICATIONS OF SECOND ORDER SYSTEM

The performance of a control system are express in terms of the transient response to a unit step input because it is easy to generate. The transient response of a control system to a unit step input depends upon the initial conditions. Consider a second order system with unit step input and the system initially at rest *i.e.*, all initial conditions are zero. The following are the common transient response characteristics.

1. Delay time (t_d)
2. Rise time (t_r)
3. Peak time (t_p)
4. Maximum overshoot (M_p)
5. Settling time (t_s)
6. Steady-state error (e_{ss})

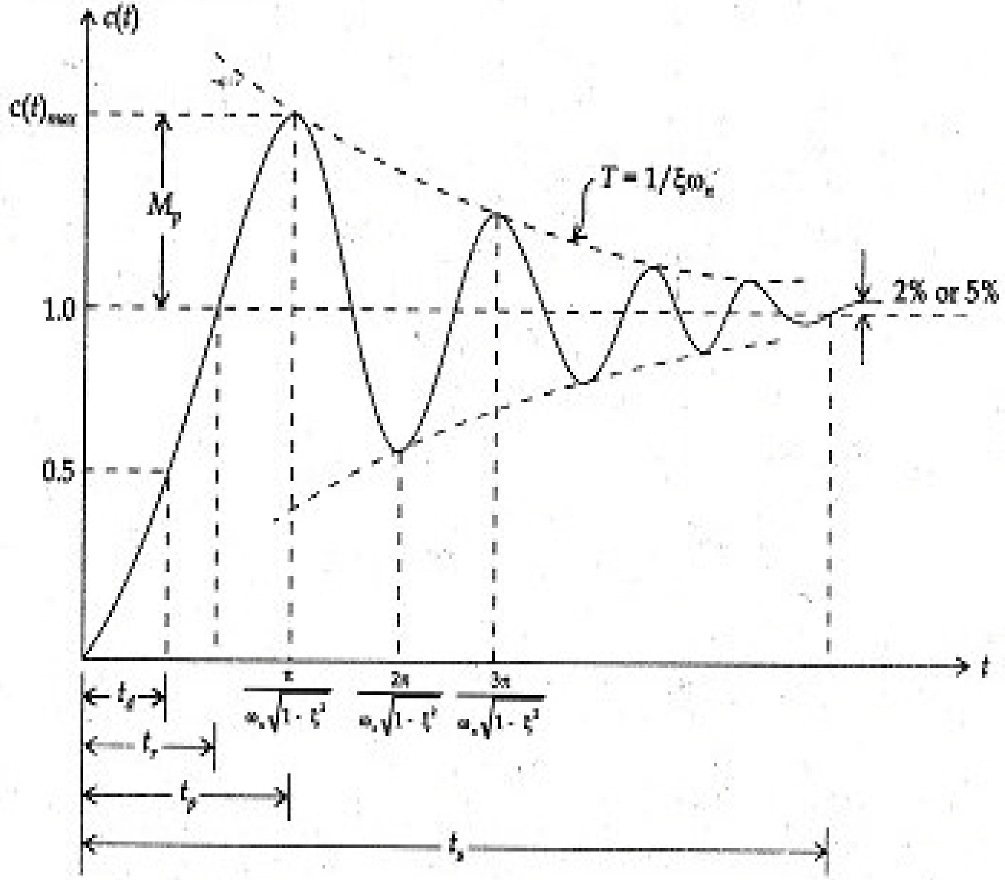


Fig. 2.17.

1. **Delay Time (t_d) :** The delay time is the time required for the response to reach 50% of the final value in first time.
2. **Rise Time (t_r) :** It is the time required for the response to rise from 10% to 90% of its final value for overdamped systems and 0 to 100% for underdamped systems.
3. **Peak Time (t_p) :** The peak time is the time required for the response to reach the first peak of the time response or first peak overshoot.
4. **Maximum Ocershoot (M_p) :** It is the normalized difference between the peak of the time response and steady output. The maximum percent overshoot is defined by

$$\text{Maximum percent overshoot} = \frac{C(t_p) - C(\infty)}{C(\infty)} \times 100$$

5. **Settling Time (t_s) :** The settling time is the time required for the response to reach and stay within the specified range (2% to 5%) of its final value.
6. **Steady State Error (e_{ss}) :** It is the difference between actual output and desired output as time 't' tends to infinity.

$$e_{ss} = \lim_{t \to \infty}[r(t) - C(t)]$$

pression for Rise Time(t_r) :

From the equation (2.20)

$$C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right]$$

where $$\phi = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$$

Let response reaches 100% of desired value. Put $c(t) = 1$

$$1 = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right]$$

or $$\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right] = 0$$

Since, $e^{-\xi\omega_n t} \neq 0$

$\therefore$ $\sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right] = 0$, or $\sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right] = \sin n\pi$

Put $n = 1$

$$\therefore \quad \left(\omega_n\sqrt{1-\xi^2}\right)t_r + \phi = \pi$$

or $$t_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\xi^2}}$$

or $$t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{\omega_n\sqrt{1-\xi^2}} \qquad \text{...(2.37)}$$

Expression for Peak Time t_p :

Since, $c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right]$

For maximum $\frac{dc(t)}{dt} = 0$

$$\frac{dc(t)}{dt} = \frac{-e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \cos\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right]\omega_n\sqrt{1-\xi^2} + \sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right]\frac{\xi\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \quad ...(2.38)$$

Since $e^{-\xi\omega_n t} \neq 0$

Equation (2.38) can be writen as

$$\cos\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right]\sqrt{1-\xi^2} = \sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right]\xi \quad ...(2.39)$$

Put $\sqrt{1-\xi^2} = \sin\phi$ & $\xi = \cos\phi$

Equation (2.39) becomes

$$\cos\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right]\sin\phi = \sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right]\cos\phi$$

or $$\sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right]\cos\phi - \cos\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right]\sin\phi = 0$$

or $$\sin\left[\omega_n\sqrt{1-\xi^2}\right]t = 0$$

the time to various peaks

$$\left(\omega_n\sqrt{1-\xi^2}\right)t_p = n\pi$$

where $n = 0, 1, 2, 3, ...$

Maximum overshoot identified by putting $n = 1$, therefore the peak time to the first overshoot

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

$$\boxed{t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}} \quad ...(2.40)$$

The first minimum (undershoot) occurs at $n = 2$

$$t_{min} = \frac{2\pi}{\omega_n\sqrt{1-\xi^2}} \quad ...(2.41)$$

Expression for Maximum Overshoot : M_p

$$C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right] \quad ...(2.42)$$

Maximum overshoot occurs at peak time *i.e.* $t = t_p$

Put $\quad t = t_p = \dfrac{\pi}{\omega_n\sqrt{1-\xi^2}}$ in equation (2.42)

$$C(t) = 1 - \frac{e^{-\xi\omega_n \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \sin\left[\omega_n\sqrt{1-\xi^2} \cdot \frac{\pi}{\omega_n\sqrt{1-\xi^2}} + \phi\right]$$

$$= 1 - \frac{e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \sin(\pi + \phi) \qquad \text{...(2.43)}$$

Since $\quad \phi = \tan^{-1}\dfrac{\sqrt{1-\xi^2}}{\xi}$ then $\quad \sin\phi = \sqrt{1-\xi^2}$ and $\sin(\pi+\phi) = -\sin\phi$

$$\therefore \quad C(t) = 1 + \frac{e^{-\pi\xi/\sqrt{1-\xi^2}}}{\sqrt{1-\xi^2}} (\sin\phi)$$

$$C(t)_{max} = 1 + e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}$$

$$\therefore \quad M_p = C(t)_{max} - 1$$

$$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}$$

$$\% M_p = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} \times 100 \qquad \text{....(2.44)}$$

Settling Time : t_s :

As shown in the Fig. 2.17, the curves for $1 \pm \dfrac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}$ are the envelope curves of the transient response for unit step input. The time constant of these envelope curves is $\dfrac{1}{\xi\omega_n}$. The speed of the decay depends upon the time constant. The settling time for a second order system is approximately four times the time constant $(1/\xi\omega_n)$

$$\therefore \quad t_s = \frac{4}{\xi\omega_n} \qquad \text{...(2.45)}$$

For overdamped system, the settling time t_s becomes large because of sluggish start. From (2.45), the settling time is inversly proportional to the product of ξ and ω_n. So, for permissible maximum overshoot, the value of ξ is known therefore the settling time can be determined by undamped natural frequency ω_n.

... the values

Error Analysis

Inside this Chapter

3.1. CLASSIFICATION OF CONTROL SYSTEM

Consider the open loop transfer function

$$G(s)H(s) = \frac{K(1+sT_1)(1+sT_2)...}{s^m(1+sT_a)(1+sT_b)+...} \quad ...(3.1)$$

In equation (3.1), the poles are at $s = -\frac{1}{T_a}$, $s = -\frac{1}{T_b}$ ... and zeros are at $s = {}^{-1}/_{T1}$, $s = {}^{-1}/_{T2}$... The equation having a term s^m in denominator, 'm' is the number of poles at the origin.

A system having no pole at origin of the 's' plane, is said to be type '0' (zero) system *i.e.*, $m = 0$

If $m = 1$ *i.e.*, 's', it means the system has a pole at origin of the s-plane and is said to be type '1' (one) system.

A system is called type '2' system if $m = 2$. and so on.

3.2. STEADY-STATE ERROR

The steady-state error is the difference between the input and output of the system during steady state. For accuracy the steady state error should be minimum.

Consider a closed loop control system shown in Fig. 3.1

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$$

or

$$E(s) = \frac{R(s)}{1+G(s)H(s)} \quad ...(3.2)$$

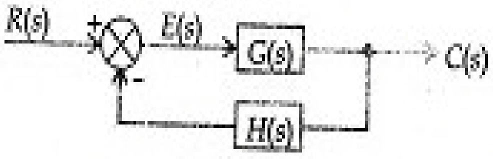


Fig. 3.1.

The steady state error of the system is obtained by applying final value theorem.

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} s.E(s) \quad ...(3.3)$$

$$e_{ss} = \lim_{s \to 0} s \cdot \frac{R(s)}{1+G(s)H(s)} \quad ...(3.4)$$

For unity feedback system $H(s) = 1$

$$e_{ss} = \underset{s \to 0}{Lim}\ s \cdot \frac{R(s)}{1+G(s)} \qquad ...(3.5)$$

From the equation (3.4) or equation (3.5) it is clear that the steady state error depends on the input and open loop transfer function.

3.3. STATIC ERROR COEFFICIENTS

(a) Static-Position Error Constant (or Coefficient) K_p

The steady state error is given by equation (3.4)

$$e_{ss} = \underset{s \to 0}{Lim}\ s \cdot \frac{R(s)}{1+G(s)H(s)}$$

For unit step input $R(s) = \frac{1}{s}$, the steady state error is given by

$$e_{ss} = \underset{s \to 0}{Lim}\ s \cdot \frac{1}{s} \frac{1}{1+G(s)H(s)} = \frac{1}{1+\underset{s \to 0}{Lim}\ G(s)H(s)}$$

$$e_{ss} = \frac{1}{1+K_p} \qquad ...(3.6)$$

K_p = static position error constant = $\underset{s \to 0}{Lim}\ G(s)H(s)$

(b) Static Velocity Error Constant (or Coefficient) Kv

$$e_{ss} = \underset{s \to 0}{Lim}\ s.R(s) \cdot \frac{1}{1+G(s)H(s)}$$

Steady state error with a unit ramp input is given by $[R(s) = 1/s^2]$

$$e_{ss} = \underset{s \to 0}{Lim}\ s \cdot \frac{1}{s^2} \frac{1}{1+G(s)H(s)}$$

$$e_{ss} = \underset{s \to 0}{Lim} \frac{1}{s+sG(s)H(s)}$$

$$= \underset{s \to 0}{Lim} \frac{1}{s\,G(s)H(s)}$$

$$e_{ss} = \frac{1}{K_v} \qquad ...(3.7$$

where $Kv = \underset{s \to 0}{Lim}\ s\,G(s)H(s)$ static velocity error coefficient.

(c) Static Acceleration Error Constant K_a

The steady-state error of the system with unit parabolic input is given by

$$R(s) = \frac{1}{s^3}$$

$$e_{ss} = \underset{s \to 0}{Lim}\ s \cdot \frac{1}{s^3} \cdot \frac{1}{1+G(s)H(s)} = \underset{s \to 0}{Lim} \frac{1}{s^2+s^2G(s)H(s)}$$

$$= \underset{s \to 0}{Lim} \frac{1}{s^2G(s)H(s)} = \frac{1}{K_a} \qquad ...(3.$$

where $K_a = \underset{s \to 0}{Lim}\ s^2\ G(s)\ H(s)$ = static acceleration constant.

3.4. STEADY-STATE ERROR FOR DIFFERENT TYPE OF SYSTEMS

1. (a) Type Zero System with Unit Step Input

$$R(s) = \frac{1}{s}$$

From equation (3.1)

$$G(s)\,H(s) = \frac{K(1+sT_1)(1+sT_2)\ldots}{(1+sT_a)(1+sT_b)\ldots} \qquad \ldots(3.9)$$

$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0}\frac{K(1+sT_1)(1+sT_2)\ldots}{(1+sT_a)(1+sT_b)\ldots} = K$$

From equation (3.6)

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+K} \qquad \boxed{e_{ss} = \frac{1}{1+K}}$$

Hence, for type zero system the static position error constant K_p is finite.

(b) Type '0' System with Unit Ramp Input

$$K_v = \lim_{s\to 0} s\cdot G(s)H(s) = \lim_{s\to 0} s\cdot\frac{K(1+sT_1)(1+sT_2)\ldots}{(1+sT_a)(1+sT_b)\ldots} = 0$$

$$\therefore \quad e_{ss} = \frac{1}{Kv} = \infty \qquad \boxed{e_{ss} = \infty}$$

(c) Type '0' System with Unit Parabolic Input

$$Ka = \lim_{s\to 0} s^2 G(s)H(s) = \lim_{s\to 0} s^2\,\frac{K(1+sT_1)(1+sT_2)\ldots}{(1+sTa)(1+sT_b)\ldots}$$

$$Ka = 0$$

$$\therefore \quad e_{ss} = \frac{1}{Ka} = \infty \qquad \boxed{e_{ss} = \infty}$$

For type '0' system, the steady state error is infinite for ramp and parabolic inputs. Hence, the ramp and parabolic inputs are not acceptable.

2(a) Type '1' System with Unit Step Input (m = 1)

$$G(s)\,H(s) = \frac{K(1+sT_1)(1+sT_2)\ldots}{s(1+sTa)(1+sT_b)\ldots}$$

$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0}\frac{K(1+sT_1)(1+sT_2)\ldots}{s(1+sT_a)(1+sT_b)\ldots} = \infty$$

$$e_{ss} = \frac{1}{1+K_p} = 0 \qquad \boxed{e_{ss} = 0}$$

(b) Type '1' System with Unit Ramp Input

$$K_v = \lim_{s\to 0} s\cdot G(s)H(s) = \lim_{s\to 0} s\cdot\frac{K(1+sT_1)(1+sT_2)\ldots}{s(1+sT_a)(1+sT_b)\ldots}$$

$$K_v = K$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K} \qquad \boxed{e_{ss} = \frac{1}{K}}$$

(c) Type '1' System with Unit Parabolic Input

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = \lim_{s\to 0} s^2 \frac{K(1+sT_1)(1+sT_2)\ldots}{s(1+sT_a)(1+sT_b)\ldots} = 0$$

$$\therefore \quad e_{ss} = \frac{1}{K_a} = \infty \qquad \boxed{e_{ss} = \infty}$$

Hence, from above relations for type '1' system, it is clear that for type '1' system step input and ramp inputs are acceptable and parabolic input is not acceptable.

3 (a) Type '2' System with Unit Step Input

$$G(s)\,H(s) = \frac{K(1+sT_1)(1+sT_2)\ldots}{s^2(1+sT_a)(1+sT_b)\ldots}$$

$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0} \frac{K(1+sT_1)(1+sT_2)\ldots}{s^2(1+sT_a)(1+sT_b)\ldots} = \infty$$

$$e_{ss} = \frac{1}{1+K_p} = 0 \qquad \boxed{e_{ss} = 0}$$

(b) Type '2' System with Unit Ramp Input

$$K_v = \lim_{s\to 0} sG(s)H(s)$$

$$= \lim_{s\to 0} s\cdot\frac{K(1+sT_1)(1+sT_2)\ldots}{s^2(1+sT_a)(1+sT_b)\ldots}$$

$$= \infty$$

$$\therefore \quad e_{ss} = \frac{1}{K_v} = 0 \qquad e_{ss} = \boxed{0}$$

(c) Type '2' System with Unit Parabolic Input

$$K_a = \lim_{s\to 0} s^2 G(s)H(s)$$

$$= \lim_{s\to 0} s^2\cdot\frac{K(1+sT_1)(1+sT_2)\ldots}{s^2(1+sT_a)(1+sT_b)\ldots} = K$$

$$\therefore \quad e_{ss} = \frac{1}{K_a} = \frac{1}{K} \qquad e_{ss} = \frac{1}{K}$$

Hence, for type '2' system all three inputs (step, ramp and parabolic) are acceptable. From Table 3.1, the diagonal elements are the finite values of steady state error.

Table 3.1.

	Type '0' System	Type '1' System	Type '2' System
Unit step input	$\frac{1}{1+k}$	0	0
Unit ramp input	∞	$\frac{1}{k}$	0
Unit parabolic input	∞	∞	$\frac{1}{k}$

EXAMPLE 3.1. The open loop transfer function of unity feedback system is given by

$$G(s) = \frac{50}{(1+0.1s)(s+10)}$$

Determine the static error coefficients K_p, K_v and K_a.

Solution :

$$K_p = \lim_{s\to 0} G(s)H(s)$$

$$= \lim_{s\to 0} \frac{50}{(1+0.1s)(s+10)} = 5$$

$$K_v = \lim_{s\to 0} s\,G(s)H(s) = \lim_{s\to 0} s\cdot\frac{50}{(1+0.1s)(s+10)} = 0$$

$$K_a = \lim_{s\to 0} s^2 \frac{50}{(1+0.1s)(s+10)} = 0$$

EXAMPLE 3.2. The forward path transfer function of a unity feedback control system is given by

$$G(s) = \frac{5(s^2+2s+100)}{s^2(s+5)(s^2+3s+10)}$$

Determine the step, ramp and parabolic error coefficients. Also determine the type of the system.

Solution :

$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0} \frac{5(s^2+2s+100)}{s^2(s+5)(s^2+3s+10)}$$

$$K_p = \infty \qquad \boxed{K_p = \infty}$$

$$K_v = \lim_{s\to 0} sG(s)H(s) = \lim_{s\to 0} s\cdot\frac{5(s^2+2s+100)}{s^2(s+5)(s^2+3s+10)}$$

$$K_v = \infty \qquad \boxed{K_v = \infty}$$

$$K_a = \lim_{s\to 0} s^2G(s)H(s) = \lim_{s\to 0} s^2\cdot\frac{5(s^2+2s+100)}{s^2(s+5)(s^2+3s+10)}$$

$$= 10 \qquad \boxed{K_a = 10}$$

In denominator the value of $m = 2$. Hence, the given system is type '2' system.

EXAMPLE 3.3. The block diagram of an electronic pacemaker is given in Fig. 3.2. Determine the steady state error for unit ramp input when K = 400 Also, determine the value of K for which the steady state error to a unit ramp will be 0.02.

Solution : Given that $K = 400$

$$R(s) = \frac{1}{s^2}$$

$$H(s) = 1$$

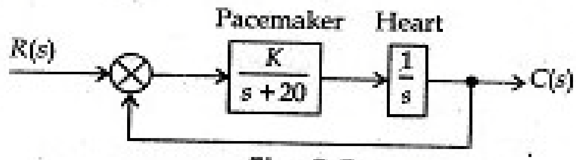


Fig. 3.2.

$$\therefore \quad G(s)\,H(s) = \frac{K}{s(s+20)}$$

Steady state error is given by $e_{ss} = \lim_{s\to 0} s\cdot\frac{R(s)}{1+G(s)H(s)}$

$$e_{ss} = \lim_{s\to 0} s \cdot \frac{1}{s^2} \frac{1}{1+\frac{K}{s(s+20)}} = \lim_{s\to 0} \frac{s+20}{s(s+20)+400} = 0.05 \quad \text{Ans.}$$

Now $e_{ss} = 0.02$ (given)

$$e_{ss} = \lim_{s\to 0} s \cdot \frac{1}{s^2} \frac{1}{1+\frac{K}{s(s+20)}}$$

$$0.02 = \lim_{s\to 0} \frac{s+20}{s(s+20)+K}$$

$$0.02 = \frac{20}{K} \qquad \therefore \qquad \boxed{K = 1000} \quad \text{Ans.}$$

EXAMPLE 3.4. For a unity feedback control system the forward path transfer function is given by

$$G(s) = \frac{20}{s(s+2)(s^2+2s+20)}$$

Determine the steady state error of the system. When the inputs are (*i*) 5 (*ii*) 5*t* (*iii*) $\frac{3t^2}{2}$.

Solution :

(*i*) $r(t) = 5 \quad \therefore \quad R(s) = \frac{5}{s}$

$$e_{ss} = \lim_{s\to 0} s \cdot E(s) = \lim_{s\to 0} s \cdot \frac{R(s)}{1+G(s)H(s)}$$

$$e_{ss} = \lim_{s\to 0} s \cdot \frac{5}{s} \cdot \frac{1}{1+\frac{20}{s(s+2)(s^2+2s+20)}} = \lim_{s\to 0} \frac{5s(s+2)(s^2+2s+20)}{s(s+2)(s^2+2s+20)+20}$$

$\therefore \quad e_{ss} = 0$

(*ii*) $R(s) = \frac{5}{s^2}$

$$e_{ss} = \lim_{s\to 0} s \cdot \frac{5}{s^2} \cdot \frac{1}{1+\frac{20}{s(s+2)(s^2+2s+20)}} = \lim_{s\to 0} s \cdot \frac{5}{s^2} \cdot \frac{s(s+2)(s^2+2s+20)}{s(s+2)(s^2+2s+20)+20}$$

$$e_{ss} = 10$$

(*iii*) $R(s) = \frac{3}{s^3}$

$$e_{ss} = \lim_{s\to 0} s \cdot \frac{3}{s^3} \cdot \frac{s(s+2)(s^2+2s+20)}{s(s+2)(s^2+2s+20)+20}$$

$$e_{ss} = \infty$$

CHAPTER-6 FEEDBACK CHARACTERISTICS OF CONTROL SYSTEM

1.32. EFFECT OF PARAMETER VARIATIONS

In control systems, the feed back reduces the error, also reduces the sensitivity of the system to parameter variations. The parameter may vary due to some change in conditions. The variation in parameter affects the performance of the system. So, it is necessary to make the system insensitive to such parameter variations.

1.32.1. Effect of Parameter Variations in Open Loop Control System

The open loop control system is shown in Fig 1.116.

Fig. 1.116.

$$\frac{C(s)}{R(s)} = G(s)$$

or,
$$C(s) = G(s) \cdot R(s) \qquad ...(1.137)$$

Let
$$\Delta G(s) = \text{Change in } G(s) \text{ due to parameter variations}$$
$$\Delta C(s) = \text{Corresponding change in output}$$

From equation (1.137)

$$C(s) + \Delta C(s) = [G(s) + \Delta G(s)]\, R(s)$$
$$C(s) + \Delta C(s) = G(s)\, R(s) + \Delta G(s)\, R(s)$$

Since,
$$G(s)\, R(s) = C(s)$$
$$C(s) + \Delta C(s) = C(s) + \Delta G(s)\, R(s)$$

or
$$\Delta C(s) = \Delta G(s)\, R(s) \qquad ...(1.138)$$

Equation (1.138) gives the change in output due to parameter variations in $G(s)$ in open loop system.

1.32.2. Effect of Parameter Variations in Closed Loop System

The closed loop system is shown in fig 1.117

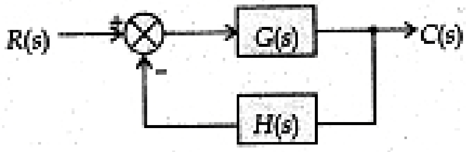


Fig. 1.117

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)\, H(s)} \qquad ...(1.139)$$

or,
$$C(s) = \frac{G(s)}{1 + G(s)\, H(s)}\, R(s)$$

$$C(s) + \Delta C(s) = \frac{G(s) + \Delta G(s)}{1 + G(s) + \Delta G(s)\, H(s)} \cdot R(s)$$

$$= \frac{G(s) + \Delta G(s)}{1 + [G(s)\, H(s) + \Delta G(s)]\, H(s)} \cdot R(s)$$

Since, $\Delta G(s)\, H(s) << [1 + G(s)\, H(s)]$, neglect $\Delta G(s)\, H(s)$

$$\therefore \quad C(s) + \Delta C(s) = \frac{G(s) + \Delta G(s)}{1 + G(s)\, H(s)}\, R(s) = \frac{G(s)}{1 + G(s)\, H(s)}\, R(s) + \frac{\Delta G(s)}{1 + G(s)\, H(s)} \cdot R(s)$$

Since,
$$C(s) = \frac{G(s)}{1 + G(s)\, H(s)}\, R(s)$$

$$C(s) + \Delta C(s) = C(s) + \frac{\Delta G(s)}{1 + G(s)\, H(s)}\, R(s)$$

or, $$\Delta C(s) = \frac{\Delta G(s)}{1+G(s)\,H(s)} R(s) \qquad ...(1.140)$$

Equation (1.140) gives the change in output due to parameter variations in $G(s)$ in a closed loop system.

Generally $[G(s)\,H(s)] >> 1$

from equation (1.140) it is clear that the change in output is reduced due to parameter variations in $G(s)$ by $[1 + G(s)\,H(s)]$. But in open loop system there is no reduction because of no feedback.

1.32.3. Effect of Feedback on Sensitivity

The parameters of any control system changes with the change in environment conditions. Also these parameters cannot be constant throughout the life. These parameter variations affects the performance of the system. For example, the resistance of the winding of a motor changes due to the change in temperature during its operation.

So, a control system should be insensitive to the parameter variations. Let P is a gain parameter that may vary due to the variations in parameters 'R' of the system. The sensitivity of the system parameter P to the parameter R is

$$S = \frac{\%\text{ change in R}}{\%\text{ change in P}}$$

$$S_P^R = \frac{d(\ln R)}{d(\ln P)} = \frac{\frac{1}{R}}{\frac{1}{F}}\frac{\partial R}{\partial P} = \frac{\partial\; R/R}{\partial\; P/P}$$

In general 'R' may be the output variable and 'P' may be the gain, the feedback factor etc.

Let $T(s)$ = Overall transfer function

$G(s)$ = Forward path transfer function is varying

Then, sensitivity will be

$$S_G^T = \frac{\partial T(s)/T(s)}{\partial G(s)/G(s)} \qquad ...(1.141)$$

For open loop system $T(s) = G(s)$

$$\therefore \quad S_G^T = \frac{\partial G(s)/G(s)}{\partial G(s)/G(s)} = 1$$

Thus, the sensitivity of open loop system is unity.

Sensitivity of closed loop system:

$$T(s) = \frac{G(s)}{1+G(s)\,H(s)} \qquad ...(1.142)$$

$$\frac{\partial T(s)}{\partial G(s)} = \frac{[1+G(s)\,H(s)]\cdot 1 - G(s)\,H(s)}{[1+G(s)\,H(s)]^2} = \frac{1}{[1+G(s)\,H(s)]^2}$$

Sensitivity is given by from equation (1.141)

$$S_G^T = \frac{G(s)}{T(s)}\cdot\frac{\partial T(s)}{\partial G(s)}$$

Put the values of $T(s)$ and $\partial T(s)/\partial G(s)$

$$= \frac{G(s)}{\frac{G(s)}{1+G(s)H(s)}} \cdot \frac{1}{[1+G(s)H(s)]^2}$$

$$S_G^T = \frac{1}{1+G(s)H(s)} \quad \text{...(1.143)}$$

From equation (1.143) the sensitivity is reduced due to the feedback by a factor $1/1 + G(s)H(s)$ as compared to open loop system.

Sensitivity due to the variation in $H(s)$:

from equation (1.142)

$$\frac{\partial T(s)}{\partial H(s)} = -\frac{[G(s)]^2}{[1+G(s)H(s)]^2}$$

$$\therefore \quad S_H^T = \frac{H(s)}{T(s)} \cdot \frac{\partial T(s)}{\partial H(s)} = \frac{H(s)}{\frac{G(s)}{1+G(s)H(s)}} \cdot \frac{-[G(s)]^2}{[1+G(s)H(s)]^2}$$

$$S_H^T = \frac{-G(s)H(s)}{1+G(s)H(s)} \quad \text{....(1.144)}$$

From equation (1.143) and (1.142) it is clear that the closed loop system is more sensitive to variations in feedback path parameters than variations in forward path variations.

1.32.4. Effect of Feedback on Overall Gain

The overall transfer function of open loop system shown in Fig. 1.118 is

$$\frac{C(s)}{R(s)} = G(s)$$

Fig. 1.118

The overall transfer function of closed loop system shown in Fig. 1.117 is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

For negative feedback the gain $G(s)$ is reduced by a factor $\frac{1}{1+G(s)H(s)}$. So due to negative feedback overall gain of the system reduces.

1.32.5. Effect of Feedback on Stability

Consider the open loop system with overall transfer function

$$G(s) = \frac{K}{s+T}$$

The pole is located at $s = -T$

Now, consider closed loop system with unity negative feedback, then overall transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{K}{s+(T+K)}$$

Now, closed loop pole is located at $s = -(T + K)$

Thus, feedback controls the time response by adjusting the location of the poles. The stability depends upon the location of poles. Thus we can say the feedback affects the stability. Feedback can improve the stability or may be harmful to stability if it is not properly design and apply.

CHAPTER-7

STABILITY CONCEPT & ROOT LOCUS

5.1. CONCEPT OF STABILITY

The concept of stability is very important to analyse and design the system. A system is said to be stable if its response cannot be made to increase indefinitely by the application of a bounded input excitation. If the output approaches towards infinite value for sufficiently large time, the system is said to be unstable.

A linear time invariant (LTI) system is stable if

1. The system is excited by a bounded input, the output is bounded (BIBO stability criteria).
2. In the absence of the input, the output tends towards zero (the equilibrium state of the system). This is known as asymptotic stable.

Consider the transfer function

$$\frac{C(s)}{R(s)} = \frac{a_m s^m + a_{m-1}s^{m-1} + \dots a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \dots + b_1 s + b_0} \quad \dots(5.1)$$

The output is given by

$$C(t) = \int_0^\infty g(\tau) r(t-\tau) d\tau \quad \dots(5.2)$$

where $g(\tau) = \mathcal{L}^{-1} G(s)$ = impulse response of the system. So, a system is said to be stable if the impulse response approaches zero for sufficiently large time. If the impulse response approaches infinity for suffciently large time, the system is said to be unstable. If the impulse response approaches a constant value for sufficiently large time, the system is said to be marginally stable.

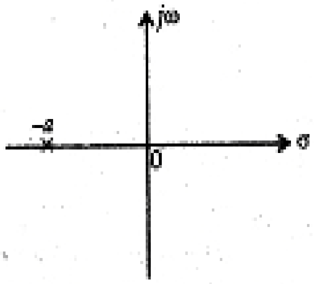
5.2. EFFECT OF LOCATION OF POLES ON STABILITY

(a) Poles on Negative Real Axis

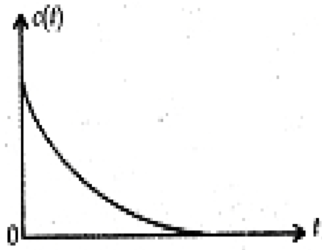
Consider a simple pole at $s = -a$ as shown in fig. 5.1*a*., the corresponding impulse response is given by

$$g(t) = £^{-1}G(s) = £^{-1}\frac{K}{s+a} = K e^{-at} \quad ...(5.3)$$

As the time '*t*' increases, the response approaches zero and the system is stable. The response is shown in fig 5.1(*b*).



(*a*) Simple plle on negative real axis



(*b*) Response

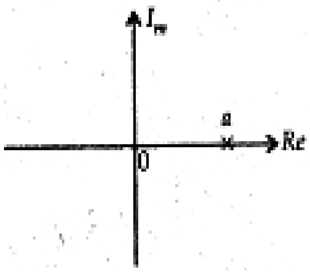
Fig. 5.1.

(b) Pole on Positive Real Axis

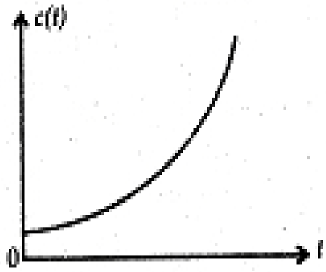
Consider a system having simple pole on positive real axis at $s = a$, the corresponding impulse response is given by

$$g(t) = £^{-1}\frac{K}{s-a} = K e^{at} \quad ...(5.4)$$

The response increases exponentially with time, hence the system is unstable. The simple pole and response are shown in Fig. 5.2 (*a*) and (*b*).



(*a*) Simple pole on positive real axis



(*b*) Response

Fig. 5.2.

(c) Pole at the Origin : Consider a Pole at Origin

$$\therefore \quad g(t) = £^{-1}\frac{K}{s} = K \quad ...(5.5)$$

This is constant value, hence the system is marginally stable. If there are two poles at the origin, the time response would be

$$g(t) = £^{-1}\frac{K}{s^2} = Kt \quad ...(5.6)$$

(a) Single pole at origin (b) Response due to single pole

Fig. 5.3.

(a) Double pole at origin (b) Impulse response

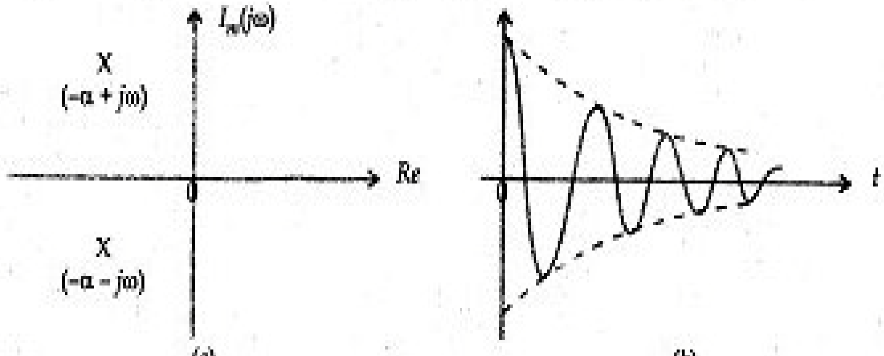
Fig. 5.4.

(d) Complex Pole in the Left Half of s-plane

Let the transfer function has a complex conjugate poles at $s = -\alpha \pm j\omega$. The time response due to the complex conjugate poles is given by

$$g(t) = £^{-1}\left[\frac{K}{s+\alpha-j\omega} + \frac{K}{s+\alpha+j\omega}\right] = £^{-1}\left[\frac{2K\,(s+\alpha)}{(s+\alpha)^2+\omega^2}\right] = 2K\,e^{-\alpha t}\cos\omega t \quad ...(5.7)$$

When t increases $g(t)$ approaches zero and the system is stable. The complex poles and corresponding time response is shown in Fig. 5.5(a) and 5.5(b) respectively.



(a) (b)

Fig. 5.5.

(e) Complex Poles in the Right Half of s-plane

Suppose the system has complex conjugate poles at $s = \alpha \pm j\omega$. The time response is given by

$$g(t) = £^{-1}\left[\frac{A}{s-\alpha-j\omega} + \frac{A}{s-\alpha+j\omega}\right] = £^{-1}\left[\frac{2A(s-\alpha)}{(s-\alpha)^2+\omega^2}\right] = 2A\,e^{\alpha t}\cos\omega t \quad ...(5.8)$$

Hence, the response increases exponentially sinusoid with time and therefore the response is unstable. The poles and time response shown in Fig. 5.6(a) and 5.6(b) respectively.

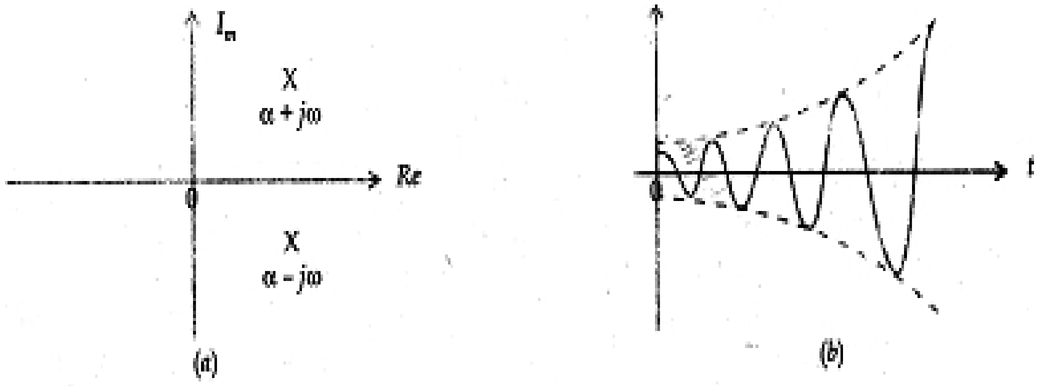


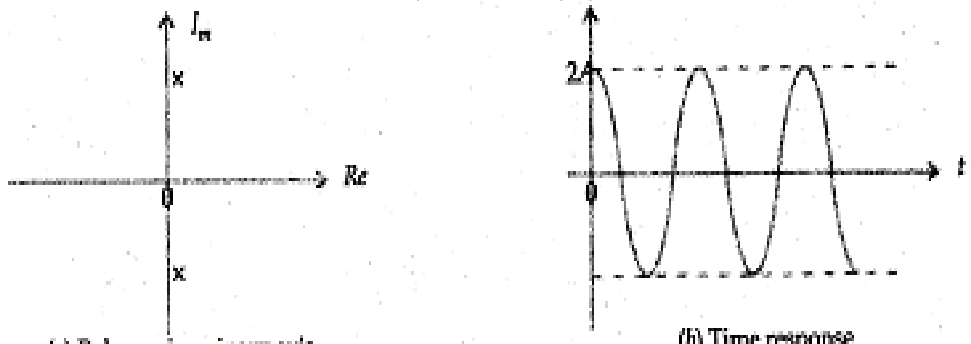
Fig. 5.6.

(f) Poles on jω-axis

If the system having the complex poles on $j\omega$-axis the corresponding time response would be

$$g(t) = £^{-1}\left[\frac{A}{s+j\omega} + \frac{A}{s-j\omega}\right] = £^{-1}\left[\frac{2As}{s^2+\omega^2}\right] = 2A\cos\omega t \qquad ...(5.9)$$

The response is marginally stable. The equation (5.9) shows the sustained oscillations of constant amplitude. This situation will also be considered unstable.



(a) Poles on imaginary axis (b) Time response

Fig. 5.7.

The overall transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \qquad ...(5.10)$$

The characteristic equation is $1 + G(s)H(s) = 0$. ...(5.11)

The necessary and sufficient condition that a feedback system be stable is that all the zeros of the characteristic equation $1 + G(s)H(s) = 0$ have negative real part. Or, in terms of poles we can say that the necessary and sufficient condition that a feedback system be stable is that all the poles of overall transfer function have negative real part.

5.3. NECESSARY BUT NOT SUFFICIENT CONDITIONS FOR STABILITY

Consider a system with characteristic equation

$$a_0 s^m + a_1 s^{m-1} + ... + a_m = 0 \qquad ...(5.12)$$

(a) All the coefficients of the equation should have same sign,

(b) There should be no missing term.

If above two conditions are not satisfied the system will be unstable. But if all the coefficients have same sign and there is no missing term we have no guarantee that the system will be stable. For stability we use Routh-Hurwitz Criterion.

5.4. THE ROUTH-HURWITZ CRITERION

Consider the following characteristic polynomial

$$a_0 s^n + a_1 s^{n-1} + \ldots\ldots + a_n = 0 \qquad \ldots(5.13)$$

where the coefficients $a_0, a_1 \ldots a_n$ are all of the same sign and none is zero.

Step 1 : Arrange all the coefficients of equation (5.13) in two rows

Row 1	a_0	a_2	a_4	....
Row 2	a_1	a_3	a_5	....

Step 2 : From these two rows form a third row

Row 1	a_0	a_2	a_4	....
Row 2	a_1	a_3	a_5	....
Row 3	b_1	b_3	b_5	....

where,

$$b_1 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}$$

$$b_3 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}$$

Step 3 : From second and third row, form a fourth row

Row 1	a_0	a_2	a_4	....
Row 2	a_1	a_3	a_5	.....
Row 3	b_1	b_3	b_5	....
Row 4	c_1	c_3	c_5	....

where,

$$c_1 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}$$

$$c_3 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_5 \\ b_1 & b_5 \end{vmatrix}$$

⋮

Step 4 : Continue this procedure of forming a new rows.

5.4.1. Statement of Routh-Hurwitz Criterion

Routh-Hurwitz criterion states that the system is stable if and only if all the elements in the first column have the same algebraic sign. If all elements are not of the same sign then the number of sign changes of the elements in first column equals the number of roots of the characteristic equation in the right half of the s-plane (or equals to the number of roots with positive real parts).

EXAMPLE 5.1. Check the stability of the system whose characteristic equation is given by

$$s^4 + 2s^3 + 6s^2 + 4s + 1 = 0$$

Solution : Obtain the array of coefficients as follows

s^4	1	6	1
s^3	2	4	
s^2	4	1	
s^1	3.5		
s^0	1		

$$b_1 = -\frac{1}{2}\begin{vmatrix} 1 & 6 \\ 2 & 4 \end{vmatrix} = 4, \qquad c_1 = -\frac{1}{4}\begin{vmatrix} 2 & 4 \\ 4 & 1 \end{vmatrix} = 3.5$$

$$b_2 = -\frac{1}{2}\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 1, \qquad d_1 = -\frac{1}{3.5}\begin{vmatrix} 4 & 1 \\ 3.5 & 0 \end{vmatrix} = 1$$

Since, all the coefficients in the first column are of the same sign (positive), the given equation has no roots with positive real parts. Hence, the system is stable.

EXAMPLE 5.2. Determine the stability of the system whose characteristic equation is given by

$$2s^4 + 2s^3 + s^2 + 3s + 2 = 0$$

Solution :

s^4	2	1	2
s^3	2	3	
s^2	–2	2	
s^1	5		
s^0	2		

$$b_1 = -\frac{1}{2}\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = -2, \qquad c_1 = -\frac{1}{(-2)}\begin{vmatrix} 2 & 3 \\ -2 & 2 \end{vmatrix} = 5$$

$$b_2 = -\frac{1}{2}\begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} = 2, \qquad d_1 = -\frac{1}{5}\begin{vmatrix} -2 & 2 \\ 5 & 0 \end{vmatrix} = 2$$

There are two changes of sign in the first column(from 2 to –2 and from –2 to 5), hence there are two roots in the right half of s-plane. The system is unstable.

EXAMPLE 5.3. Determine the stability of the system having following characteristic equation

$$2s^4 + 5s^3 + 5s^2 + 2s + 1 = 0$$

Solution :

s^4	2	5	1
s^3	5	2	
s^2	4.2	1	
s^1	0.809		
s^0	1		

From the above Routh table :

No. of sign changes in first column = 0

No. of poles on the right hand side of s-plane = 0

Hence, the system is stable.

EXAMPLE 5.4. Check the stability of the system having following characteristic equation.

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

Solution :

s^4	1	3	5
s^3	2	4	
s^2	1	5	
s^1	–6		
s^0	5		

$$b_1 = -\frac{1}{2}\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 1, \qquad b_2 = -\frac{1}{2}\begin{vmatrix} 1 & 5 \\ 2 & 0 \end{vmatrix} = 5$$

$$c_1 = -\frac{1}{1}\begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix} = -6, \qquad d_1 = -\frac{1}{(-6)}\begin{vmatrix} 1 & 5 \\ -6 & 0 \end{vmatrix} = 5$$

From above table :

No. of sign changes in first column = 2

No. of roots in right half of s-plane = 2

Hence, the system is unstable.

EXAMPLE 5.5. **A closed loop control system has the characteristic equation given by** $s^3 + 4.5s^2 + 3.5s + 1.5 = 0$

Investigate the stability using Routh-Hurwitz criterion. *(R.M.L. University Faizabad, 2001)*

Solution :

s^3	1	3.5
s^2	4.5	1.5
s^1	3.17	
s^0	1.5	

No. of sign changes in first column = 0

No. of roots in right half of s-plane = 0

Hence, system is stable.

EXAMPLE 5.6. **Check the stability of the system, having following characteristic equation.**

Solution: $s^5 + 6s^4 + 3s^3 + 2s^2 + s + 1 = 0$

s^5	1	3	1
s^4	6	2	1
s^3	2.67	0.83	
s^2	0.135	1	
s^1	−18.95		
s^0	1		

No. of sign change in first column = 2

No. of poles on right half of s-plane = 2

Hence, system is unstable.

SPECIAL CASES

Case 1 : If a first column term in any row is zero, but the remaining terms are not zero or there is no remaining term, then multiply the original equation by a factor $(s + a)$ where 'a' is any positive real number. The simplest value of 'a' is 1 (take $a = 1$). Consider the following example.

EXAMPLE 5.7. **Investigate the stability**

$s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$

Solution :

s^5	1	2	3
s^4	1	2	5
s^3	0		
s^2			
s^1			
s^0			

Now, multiply the characteristic equation by $(s + 1)$

$(s + 1)(s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5) = 0$

or, $s^6 + 2s^5 + 3s^4 + 4s^3 + 5s^2 + 8s + 5 = 0$

s^6	1	3	5	5
s^5	2	4	8	
s^4	1	1	5	
s^3	2	−2		
s^2	2	5		
s^1	−7			
s^0	1			

From the above table :

No. of sign change in the first column = 2

No. of roots in the right half of s-plane = 2

Hence, system is unstable.

3.6 Root locus technique

3.6.1 *Introduction*

The main aim of a control system Engineer is to design a control system that meets the desired specifications. While designing the control system, it is important to determine whether his design of control system meets the desired specifications or not. This can be done by observing the response of the system for test signals. If the response does not meet the specifications, then he has to redesign the system by changing the parameters. Moreover the designer must also check the stability of the system. This can be found by determining the roots of the characteristics equation $1 + G(s)H(s) = 0$. For higher order systems, it is a laborious process. But applying Routh criterion to the characteristic equation it is possible to determine whether the system is stable or unstable. But is has certain drawbacks.

1. It does not give sufficient information about relative stability of the system, that is, the amount of overshoot and the settling time etc. Sometimes poor relative stability may bring the system to the verge of instability.
2. It does not help much in design problem in which the designer is required to achieve the desired performance by varying one or more system parameters.

The desired behavior of the system is specified in terms of steady state error, peak over shoot, settling time, rise time etc., for a step input. In section (2.19), we studied that the location of closed loop system poles (roots of characteristic equation) and the transient response specifications are interrelated. It is frequently necessary to adjust one or more system parameters in order to obtain suitable locations of roots. Therefore it is worthwhile to determine how the roots of the characteristic equation of a given system migrate on the s-plane as one of the parameters is adjusted. The locus of this migration is known as root locus. Once the locus is obtained, one can select the poles on the root locus which meet the desired specifications and then can obtain the corresponding adjustable parameter. That is, by adjusting the location of closed loop pole one can obtain the desired specifications.

Construction Rules

Rule 1:

The root locus is symmetrical about the real axis and the number of branches equal to the order of the polynomial (Number of poles of the open loop transfer function).

The roots of the characteristic equations are either real, imaginary or complex conjugate or combination of the above; therefore the root locus is symmetrical about real axis. The root locus above the real axis is mirror image of the root locus below the real axis and vice versa. The number of branches of the root locus is equal to the order of the characteristic polynomial.

Rule 2:

All branches of root locus starts at open loop poles (when $k = 0$) and ends at either open loop zero or infinity (when $k = \infty$). The number of branches terminating at infinity equals to the difference between the number of poles and number of zeros.

Rule 3:

A point on the real axis lies on the root locus if the sum of the poles and zeros on the real axis to the right of the point is an odd number.

Consider the open loop pole and zero configuration as shown in Fig. 3.23. Let s_0 be the test point. To check whether the test point s_0 is on the root locus or not, join all the poles and zeros to this point. At this point the angles made by the lines joining p_1, p_2, p_3 and p_4 with s_0 are $\angle s_0 + p_1, \angle s_0 + p_2, \angle s_0 + p_3$ and $\angle s_0 + p_4$ respectively. Similarly the angles made by the lines joining z_1, z_2, z_3 and z_3' are $\angle s_0 + z_1, \angle s_0 + z_2, \angle s_0 + z_3$ and $\angle s_0 + z_3'$ respectively. From Fig. 3.23 we can observe that the angles made by p_1, p_2 and z_1 are

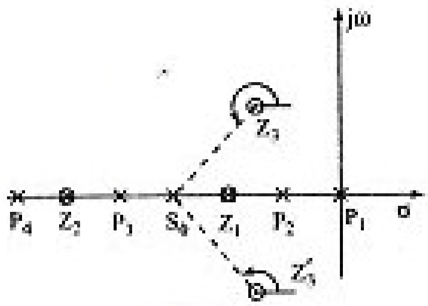


Fig. 3.23

$$\angle s_0 + p_1 = \angle s_0 + p_2 = \angle s_0 + z_1 = 180^\circ \tag{3.33}$$

and the angle made by z_2, p_3 and p_4 are

$$\angle s_0 + z_2 = \angle s_0 + z_3 = \angle s_0 + p_4 = 0^\circ \tag{3.34}$$

Angle made by $z_3$and $z_3^1$with s_0 are equal and opposite (ie) $\angle s_0 + z_3 + \angle s_0 + z_3^1 = 0$. Therefore it is not necessary to consider complex poles and zeros.

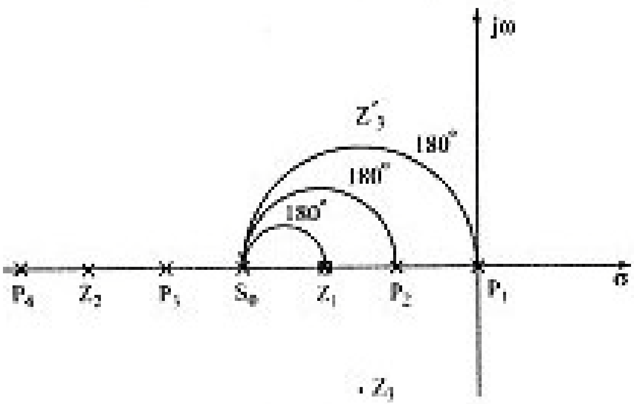


Fig. 3.24

The sum of the angles contributed by complex conjugate poles is zero. From this we conclude the following

1. The angle contribution of all the poles and zeros on the real axis to the right of the point is 180°.
2. The angle contribution of all the poles and zeros on the real axis to the left of the test point is 0°.

3. The angle contribution by complex conjugate poles and zeros is zero.

From Eq. (3.33) & Eq. (3.34) the angle of $G(s)H(s)$ with the point s_0 is given by

$$\Phi_{p1} + \Phi_{p2} + \Phi_{p3} + \Phi_{p4} + \Phi_{z1} + \Phi_{z2} + \Phi_{z3} + \Phi_{z3'}$$
$$= 180^\circ + 180^\circ + 0^\circ + 180^\circ + 0^\circ + 0^\circ + 0^\circ + 0^\circ = 180^\circ$$

180° is odd multiple of 180° therefore s_0 is a point on the locus.

Similarly, For the test point s_1 the net angle contribution to all open loop poles and zeros are given by

$$\Phi_{p1} + \Phi_{p2} + \Phi_{p3} + \Phi_{p4} + \Phi_{z1} + \Phi_{z2} + (\Phi_{z3} + \Phi_{z3'})$$
$$= 180^\circ + 180^\circ + 180^\circ + 0^\circ - 180^\circ + 0^\circ + 0^\circ + 0^\circ = 360^\circ \neq \pm 180^\circ(2q+1)$$

Therefore s_1 is not a point on the root locus. Thus the necessary condition for determining the real axis locus is

$$(n_z - n_p)180^\circ = \pm(2q+1)180^\circ$$

Where n_p is the number of poles on the real axis to the right of the test point and is the number of zeros on the real axis to the right of the test point. Eq. (3.25) satisfies when $n_z - n_p$ must be an odd number. If $n_z - n_p$ is an odd number then $n_p + n_z$ also an odd number. Therefore we can conclude that if the total number of poles and zeros to the right of the test point s_0 on the real axis is odd then the test point lies on the root locus.

Example 3.13. Draw the root locus for the unity feedback system with open loop transfer function

$$G(s) = \frac{k(s+1)(s+3)}{s(s+2)(s+4)}$$

Solution .

The three rules so far we have seen are sufficient to draw the root locus of the given system.

Step 1. The number of open loop poles are three. Therefore the number of branches of the root locus are three. The plot of poles and zeros are shown in Fig. 3.25

Step 2. The three branches of the root locus starts from the open loop poles $s = 0, -2, -4$. Out of these three branches two branches of the root locus terminate at the two open loop zeros and one branch terminates at infinity.

Step 3. All the points between 0 and -1, -2 and -3, -4 and $-\infty$ lie on the root locus for which the sum of open loop poles and zeros to the right of test points are 1,3 and 5 respectively (all points are having odd number of poles and zeros to its right).

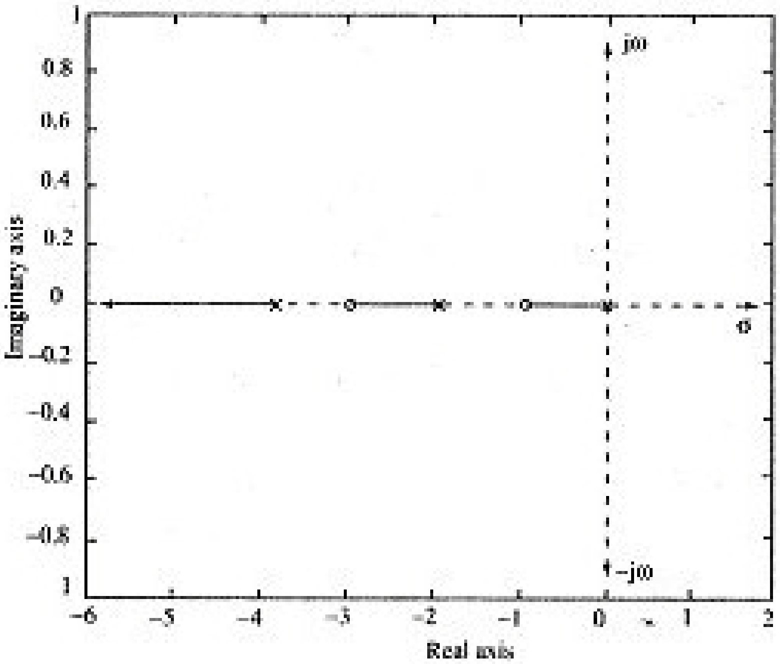


Fig. 3.25

Rule 4:

The $(n - m)$ root locus branches that proceed to infinity do so along the asymptotes, with angles

$$\Phi_A = \frac{(2q+1)180^\circ}{n-m} \quad q = 0, 1, 2, \ldots\ldots (n-m-1)$$

Consider a test point s_0 at infinity, the angles made by the line joining the test point s_0 and the open loop poles and zeros are equal to each other (say ϕ_A^o). The total number of such angles is equal to $n - m$. Therefore the total angle made by the test point s_0 with all open loop poles and zeros is equal to $(n-m)\phi_A^o$. This angle must satisfy the angle criterion $(n-m)\phi_A^o = \pm 180^\circ(2q+1)$

$$(n-m)\Phi_A^o = (2q+1)180^\circ \tag{3.35}$$

$$\Phi_A^o = \frac{(2q+1)180^\circ}{(n-m)} \tag{3.36}$$

where $q = 0, 1, 2, 3 \ldots (n-m-1)$, since $(n-m)$ branches of the root locus tends to infinity along the asymptotes, the number of asymptotes is equal to $n - m$. Therefore q varies from 0 to $n - m - 1$.

$$\rightarrow \Phi_A^0 = \frac{(2q+1)180^\circ}{(n-m)} \quad q = 0, 1, 2, 3 \ldots n - m - 1. \tag{3.37}$$

$$\Rightarrow \Phi_A^0 = \frac{(2q+1)180^\circ}{(\text{number of poles} - \text{number of zeros})} \quad q = 0, 1, 2, 3 \ldots n - m - 1. \tag{3.38}$$

Step 3. All the points between 0 and -2, -3 and -4 lie on the root locus since the sum of poles and zeros to the right of these points is odd (1 and 3 respectively).

Step 4. The two root locus branches that proceed to infinity do so along the asymptotes with angles

$$\phi_A = \frac{(2q+1)180^\circ}{n-m}\quad q = 0,1,2,\ldots\ldots(n-m-1).$$

$$\phi_A = \frac{(2q+1)180^\circ}{2}; q = 0,1$$

$$= 90^\circ, 270^\circ$$

Step 5. The centroid, the point of intersection of the asymptotes on the real axis is given by

$$\sigma_A = \frac{\text{Sum of real parts of poles} - \text{Sum of real parts of zeros}}{\text{Number of poles} - \text{Number of zeros}}$$

$$\sigma_A = \frac{(-2-3-4)-(-1)}{3-1} = \frac{-8}{2} = -4$$

Step 6. The break away points of the root locus are the solution of $\frac{dk}{ds} = 0$

$$G(s)H(s) = \frac{k(s+1)}{(s+2)(s+3)(s+4)}$$

$$k = -\frac{(s+2)(s+3)(s+4)}{(s+1)}$$

$$= \frac{-(s^3+9s^2+26s+24)}{(s+1)}$$

$$\frac{dk}{ds} = \frac{(s+1)(3s^2+18s+26)-(s^3+9s^2+26s+24)}{(s+1)^2}$$

$$(3s^3+21s^2+44s+26)-(s^3+9s^2+26s+24) = 0$$

$$2s^3+12s^2+18s+2 = 0$$

$$s^3+6s^2+9s+1 = 0$$

The roots are

$$-3.5321, -2.3473, -0.1206$$

The root -3.5321 alone lies on the root locus. Hence the break away point is at -3.5321.

Rule-6

The breakaway points on which multiple roots of the characteristics equation occur of the root locus are the solution of dk/ds =0

The complete root locus plot is shown in Fig. 3.27.

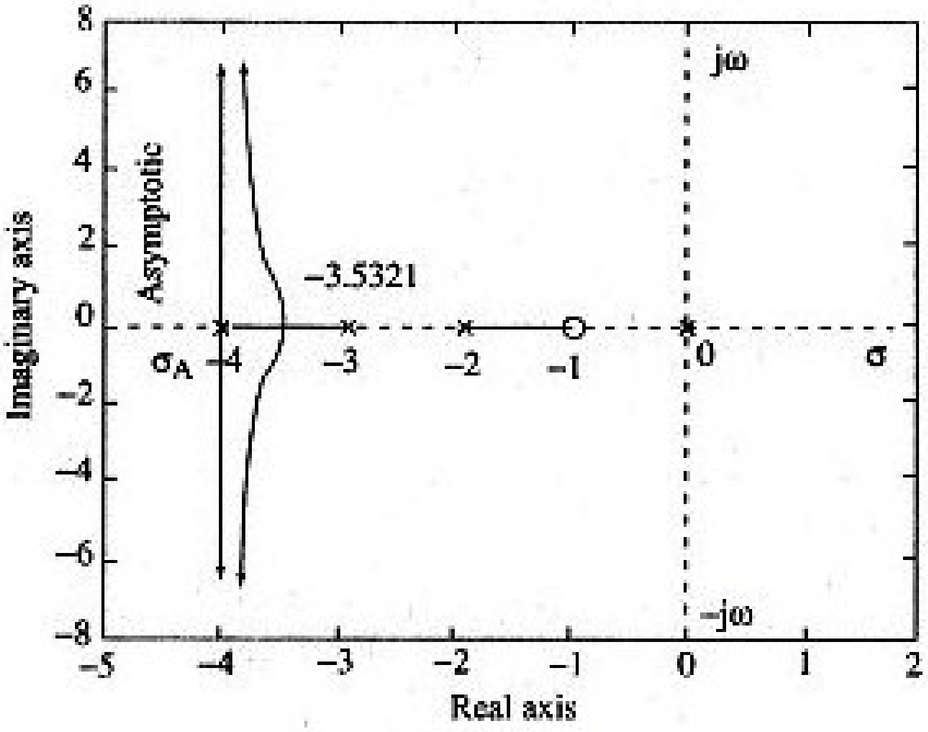


Fig. 3.27

Rule 7:

The angle of departure from an open loop pole is given by

$$\phi_p = \pm 180°(2q+1) + \phi; \quad q = 0, 1, 2 \ldots$$

where ϕ is net angle contribution to this pole by all other open loop poles and zeros. Similarly the angle of arrival at an open loop zero is given by

$$\phi_z = \pm 180°(2q+1) - \phi; \quad q = 0, 1, 2 \ldots\ldots$$

where ϕ is the net angle contribution to the zero under consideration by all other open loop poles and zeros.

Example 3.16. Sketch the root locus for a system with open loop transfer function

$$G(s)H(s) = \frac{k(s+1)}{s^2+4s+13}$$

Solution .

The open loop poles are at $s_{1,2} = -2 \pm j3$ and open loop zero is at $s_3 = -1$. That is $n = 2; m = 1$.

Step 1. There are two root locus branches since the system has two open loop poles.

Step 2. The two branches of the root locus starts at open loop poles at $-2 \pm j3$. One branch terminates at open loop zero (Since it has only one zero) and the other terminates at infinity when $k = \infty$.

Step 3. All the points on the real axis between $-\infty$ and -1 are on the root locus branch.

Step 4. The root locus branch that terminates at infinity do so along the asymptote with angle

$$\begin{aligned}\phi_A &= \frac{(2q+1)180^\circ}{n-m};\ q = 0, 1, 2 \ldots\ldots n-m.\\ &= \frac{(2q+1)180^\circ}{1};\ q = 0\\ &= 180^\circ\end{aligned}$$

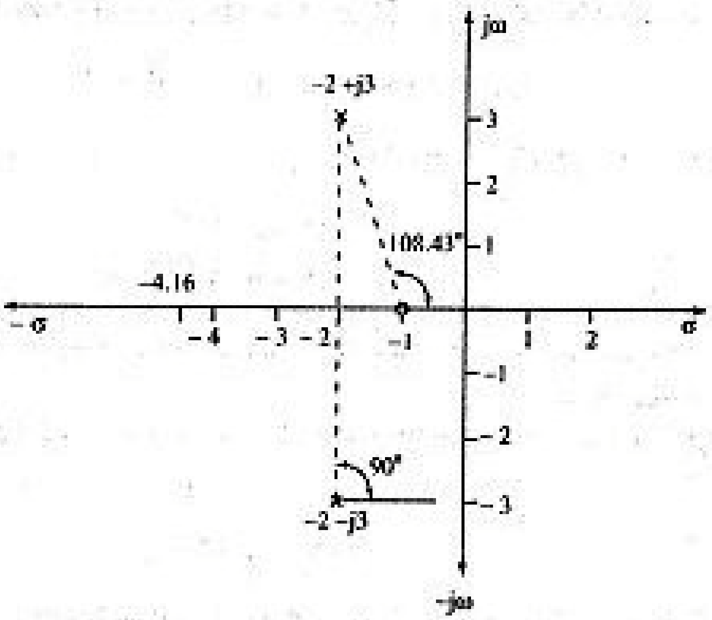


Fig. 3.29

Step 5. The break away points of the root locus are the solution of $\dfrac{dk}{ds} = 0$

$$k = \frac{-(s^2 + 4s + 13)}{s+1}$$

$$\frac{dk}{ds} = -\left[\frac{(s+1)(2s+4) - (s^2+4s+13)}{(s+1)^2}\right] = 0$$

$$(2s^2 + 6s + 4) - (s^2 + 4s + 13) = 0$$
$$\Rightarrow \quad s^2 + 2s - 9 = 0$$

The roots are at $s_1 = -4.16$ and $s_2 = 2.16$.

The break away point is at -4.16 since this point is on the root locus but the other root 2.16 is not on the root locus.

Step 6. The angle of departure from an open loop pole is given by

$$\phi_p = \pm 180°(2q+1) + \phi; q = 0, 1, 2$$

For $q = 0$

$$\phi_p = \pm 180° + \phi$$

where ϕ is the net angle contribution at this pole due to the other open loop poles and zeros.

Let us consider the pole at $-2 + j3$.

The net angle contribution $\phi = \phi_{z1} - \phi_{p2} = 108.43° - 90° = 18.43°$.

The angle of department at pole p_1 is

$$\phi_p = \pm 180° + 18.43°$$
$$= 198.43°, \quad 161.57°$$

The complete root locus plot is shown in Fig. 3.30.

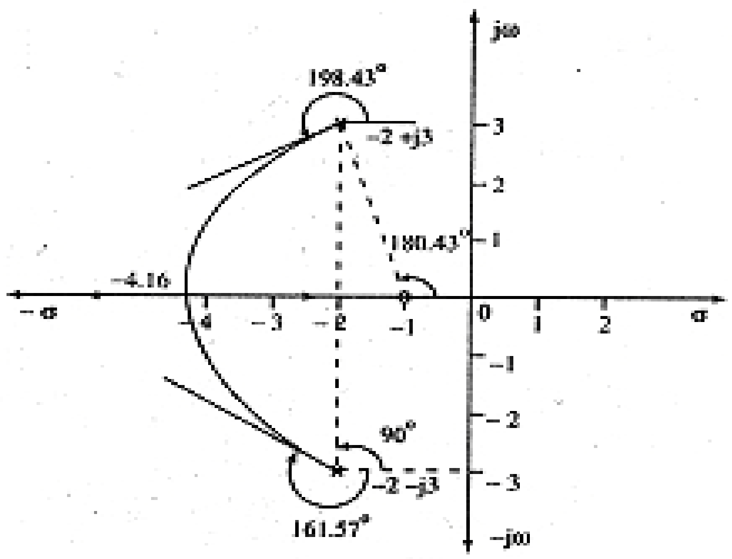


Fig. 3.30

Rule 8:

The intersection of root locus with imaginary axis can be determined using the Routh criterion.

Rule 9:

The open loop gain k (Transfer function in pole-zero form) at any point s_0 on the root locus is given by

$$k = \frac{\prod |(s + p_i)|}{\prod |(s + z_i)|}$$

$$k = \frac{\text{Product of phasor lengths from } s_0 \text{ to open loop poles}}{\text{Product of phasor lengths from } s_0 \text{ to open loop zeros}}$$

Example 3.17. Sketch the root locus of a feedback system whose open loop transfer function is given by

$$G(s)H(s) = \frac{k}{s(s+2)(s+3)}$$

Solution .

Using the rules discussed so far we can sketch the root locus. The open loop poles are at $s = 0, -2$ and -3 and there is no open loop zeros.

Step 1. The numbers of root locus branches are three since the number of open loop poles are three.

Step 2. The three branches of the root locus originate from the open loop poles at $s = 0, -2$ and -3 when $k = 0$ and all the three branches terminate at infinity when $k = \infty$.

Step 3. All the points between 0 and -2, -3 and $-\infty$ lies on the root locus for which the sum of open loop poles and zeros to the right of the test point are 1 and 3 respectively.

Step 4. The three root locus branches that proceed to infinity do so along the asymptotes with angles

$$\phi_A = \frac{(2q+1)180^\circ}{3}; q = 0, 1, 2$$

$$\phi_A = 60^\circ, 180^\circ, 300^\circ$$

Step 5. The centroid, the point of intersection of the asymptotes on the real axis is given by

$$\sigma_A = \frac{\text{Sum of real part of poles} - \text{Sum of real part of zeros}}{\text{Number of poles} - \text{Number of zeros}}$$

$$= \frac{0 - 2 - 3 - 0}{3} = \frac{-5}{3} = -1.667$$

Step 6. The break away points of the root locus are the solution of $\frac{dk}{ds} = 0$

$$G(s)H(s) = \frac{k}{s(s+2)(s+3)}$$
$$1 + G(s)H(s) = 0$$
$$\Longrightarrow G(s)H(s) = -1$$
$$\frac{k}{s(s+2)(s+3)} = -1$$
$$k = -s(s+2)(s+3) = -(s^3 + 5s^2 + 6s)$$
$$\frac{dk}{ds} = -[3s^2 + 10s + 6] = 0$$

The roots of $\frac{dk}{ds} = 0$ are -2.5485 and -0.7847 The point $s = -2.5485$ is not on the root locus. Therefore the breakaway point is -0.7847 which is on the root locus.

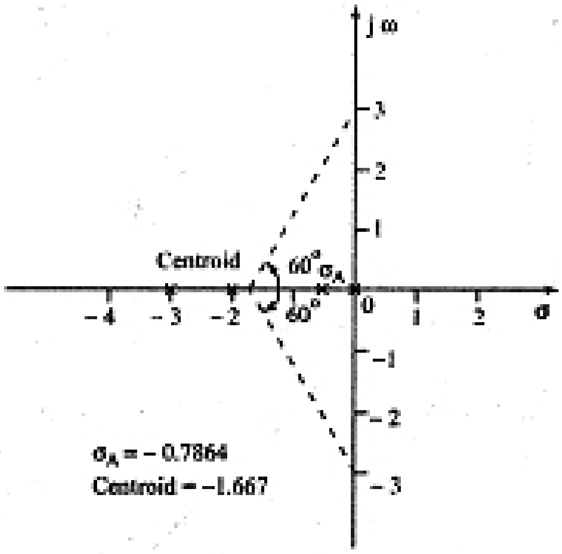


Fig. 3.31

Step 7. The intersection of the root locus with imaginary axis can be determined using Routh criterion. The characteristic equation is given by

$$1 + G(s)H(s) = 0$$

$$1 + \frac{k}{s(s+2)(s+3)} = 0$$
$$s(s+2)(s+3) + k = 0$$
$$s^3 + 5s^2 + 6s + k = 0$$

s^3	1	6
s^2	5	k
s	$(30-k)/5$	0
s^0	k	

$$(30-k)/5 \geq 0$$
$$0 < k < 30$$
$$s^2 = \pm j\sqrt{6}$$
$$s = \pm j2.449,$$

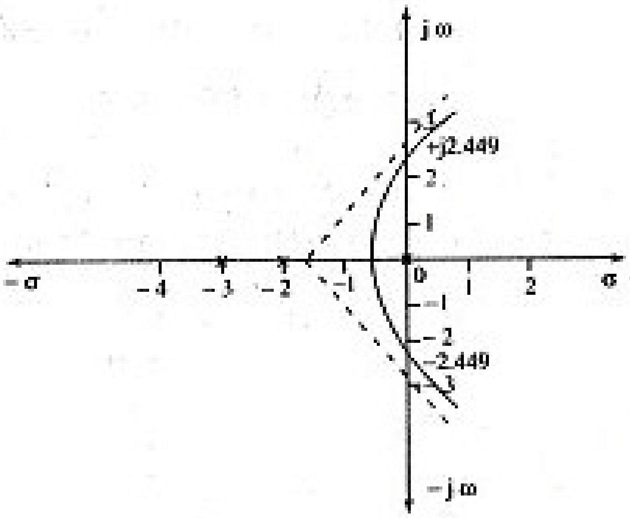


Fig. 3.32

where s is the point at which the root locus crosses imaginary axis. The complete root locus is shown in Fig. 3.32.

Example 3.18. Sketch the root locus for a unity feedback system with open loop transfer function

$$G(s) = \frac{k}{s(s^2+8s+32)}$$

Solution .
The poles of the open loop transfer function are the roots of the denominator

$$s(s^2+8s+32) = 0$$
$$\Longrightarrow P_1 = 0$$
$$P_{2,3} = \frac{-8 \pm \sqrt{64-4(32)}}{2} = \frac{-8 \pm \sqrt{-64}}{2} = -4 \pm j4$$

Mark the poles with × symbol on the graph sheet.

Step 1. There are three open loop poles, hence the number of branches in the root locus are three and no zeros.

Step 2. The three branches starts at $p_1 = 0, p_2 = -4 + j4$ and $p_3 = -4 - j4$ when $k = 0$ and terminate at infinity when $k = \infty$.

Step 3. All the points on the real axis between $-\infty$ to 0 lie on the root locus, since there is one pole to the right of these points.

Step 4. The three branches that terminates at infinity do so along the asymptotes with angles

$$\phi_n = \frac{(2q+1)180^\circ}{n-m} \quad q = 0, 1, 2, \ldots (n-m-1)$$

$$= \frac{(2q+1)180^\circ}{3} \quad q = 0, 1, 2$$

For $q = 0$ $\phi_{A1} = \frac{180^\circ}{3} = 60^\circ$

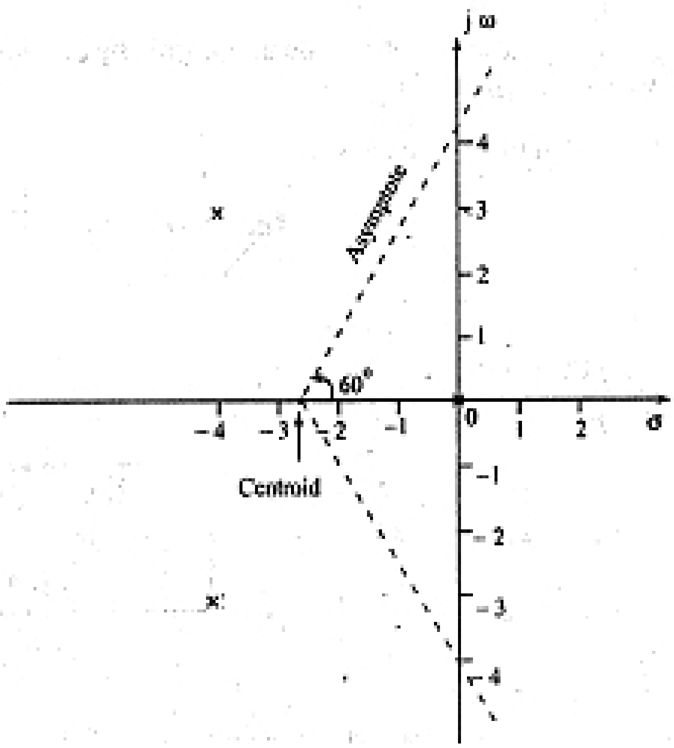
For $q = 1$ $\phi_{A2} = \frac{3(180^\circ)}{3} = 180^\circ$

For $q = 2$ $\phi_{A3} = \frac{5(180^\circ)}{3} = 300^\circ$

Step 5. The asymptotes meet at a point known as centroid

$$\sigma_A = \frac{\text{Sum of real parts of poles - Sum of real parts of zeros}}{\text{Number of poles - Number of zeros}}$$

$$= \frac{-4-4-0}{3} = -\frac{8}{3} = -2.667$$



Mark the centroid on the real axis and draw the asymptotes with angles calculated in step 4 using protractor.

Step 6. The break away point of root locus are the solution of $\frac{dk}{ds} = 0$

$$G(s)H(s) = \frac{k}{s(s^2+8s+32)}, \quad H(s) = 1$$

we know

$$1 + G(s)H(s) = 0$$
$$\Rightarrow \quad 1 + \frac{k}{s(s^2+8s+32)} = 0$$
$$k = -s(s^2+8s+32)$$
$$\frac{dk}{ds} = 0$$
$$\Rightarrow 3s^2 + 16s + 32 = 0$$

The roots are $\dfrac{-8 \pm j4\sqrt{2}}{3}$

The points are not on the root locus. Therefore there is no breakaway point.

Step 7. The angle of departure ϕ_p of a root locus from a complex open loop pole is

$$\phi_p = 180^\circ + \phi$$

when ϕ is the net angle contribution at this pole by all other open loop poles and zero as shown in Fig. 3.34.

The angle of departure at pole p_2 is

$$\phi_{p_2} = 180^\circ + \phi$$

where

$$\begin{aligned}\phi &= -135^\circ - 90^\circ \\ &= -225^\circ\end{aligned}$$

$$\phi_{p_2} = 180^\circ - 225^\circ = -45^\circ$$

$\tan^{-1}\left(\frac{4}{4}\right) = 45^\circ$
$\phi_{p2} = 180^\circ - 45^\circ = 135^\circ$
$\phi_{p3} = 90^\circ$

Similarly

$$\phi_{p_3} = -\phi_{p_2} = -(-45^\circ) = 45^\circ$$

Using protractor mark the angle of departure of complex pole

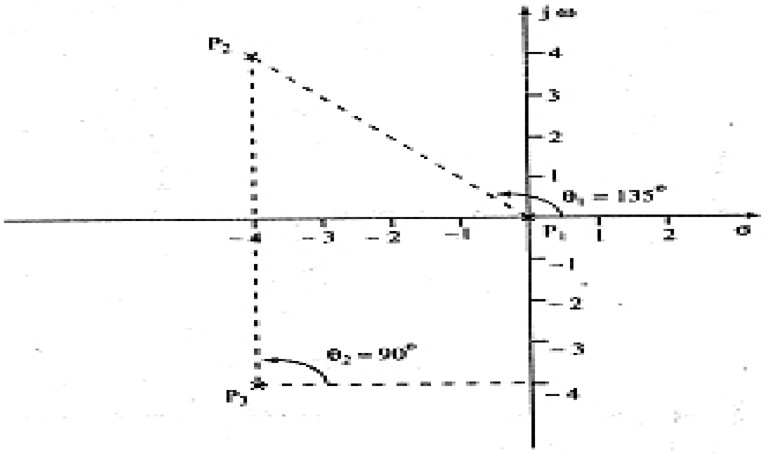


Fig. 3.34

Step 8. The crossing point on the imaginary axis can be found using Routh criterion.

The characteristic equation is given by

$$1 + G(s)H(s) = 0$$

$$1 + \frac{k}{s(s^2 + 8s + 32)} = 0$$

$$s^3 + 8s^2 + 32s + k = 0$$

s^3	1	32	0
s^2	8	k	0
s^1	$\frac{256 - k}{8}$	0	
s^0	k		

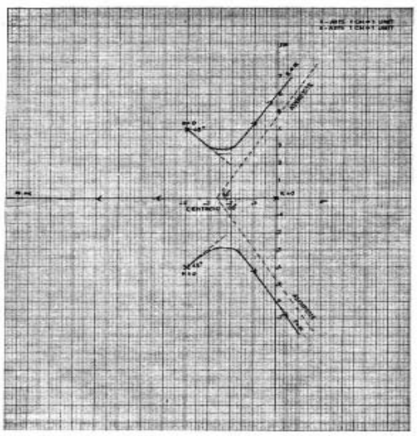
For stability

$\frac{256 - k}{8} > 0$ and $k > 0$

$\Rightarrow \quad 0 < k < 256$

When $k = 256$, the root locus crosses the imaginary axis. The auxiliary equation is $8s^2 + k = 0 \Rightarrow 8s^2 + 256 = 0$. $\therefore s = \pm j\sqrt{32}$.

The complete root locus plot is shown in Fig. 3.35



Frequency-response analysis&Bode Plot

4.4. POLAR PLOT

The polar plot of a sinusoidal transfer function $G(j\omega)$ is a plot of the magnitude of $G(j\omega)$ versus the phase angle of $G(j\omega)$ on polar coordinates as 'ω' is varied from zero to infinity. The polar plot, therefore is the locus of vectors $|G(j\omega)| \angle G(j\omega)$ as ω is varied from zero to infinity. Thus we can express the frequency response function $G(j\omega)$ in the polar form $Me^{j\phi(\omega)}$ and plot the vector $Me^{j\phi(\omega)}$ in the G-plane as 'ω' varies from zero to infinity. In polar plots the magnitude of $G(j\omega)$ is plotted as the distance from the origin while the phase angle is measured from positive real axis. Positive phase angle is measured counter clockwise while negative phase angle is measured clockwise from the positive real axis. The polar plot is often called the Nyquist Plot.

The advantage in using a polar plot that it depicts the frequency response characterstics of a system over the entire frequency range in a single plot. The disadvantage is that the plot does not indicate the contributions of each individual factor of the open loop transfer function.

4.5. PROCEDURE TO SKETCH THE POLAR PLOT

Step 1 : Determine the transfer function $G(s)$ of the system.

Step 2 : Put $S = j\omega$ in the transfer function to obtain $G(j\omega)$

Step 3 : At $\omega = 0$ and $\omega = \infty$ calculate $|G(j\omega)|$. by $\lim_{\omega \to 0} |G(j\omega)|$ and $\lim_{\omega \to \infty} |G(j\omega)|$.

Step 4 : Calculate the phase angle of $G(j\omega)$ at $\omega = 0$ and $\omega = \infty$ by $\lim_{\omega \to 0} \angle G(j\omega)$ and $\lim_{\omega \to \infty} \angle G(j\omega)$

Step 5 : Rationalize the function $G(j\omega)$ and separate the real and imaginary parts.

Step 6 : Equate the imaginary part $I_m |G(j\omega)|$ to zero and determine the frequencies at which plot intersects the real axis and caluculate the value $G(j\omega)$ at the point of intersection by substituting the determined value of frequency in the expression of $G(j\omega)$.

Step 7 : Equate the real part $\text{Re}\,|G(j\omega)|$ to zero and determine the frequencies at which plots intersects the imaginary axis and calculate the value of $G(j\omega)$ at the point of intersection by substituting the determined value of frequency in the rationalized expression of $G(j\omega)$.

Step 8 : Sketch the polar plot with the help of above information.

1. TYPE 'ZERO' SYSTEM

$$G(s) = \frac{K}{(1+ST_1)(1+ST_2)}$$

Step 1 : Put $S = j\omega$

$$G(j\omega) = \frac{K}{(1+j\omega T_1)(1+j\omega T_2)}$$

$$G(j\omega) = \frac{K}{\sqrt{1+(\omega T_1)^2}\,\sqrt{1+(\omega T_2)^2}} \angle -\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

Step 2 : Taking the limit for the magnitude of $G(j\omega)$.

$$\lim_{\omega \to 0} |G(j\omega)| = \lim_{\omega \to 0} \frac{K}{\sqrt{1+(\omega T_1)^2}\,\sqrt{1+(\omega T_2)^2}} = K$$

$$\lim_{\omega \to \infty} |G(j\omega)| = \lim_{\omega \to \infty} \frac{K}{\sqrt{1+(\omega T_1)^2}\,\sqrt{1+(\omega T_2)^2}} = 0$$

Step 3 : Taking the limit for the phase angle of $G(j\omega)$

$$\lim_{\omega \to 0} \angle G(j\omega) = \lim_{\omega \to 0} \angle -\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 = 0$$

$$\lim_{\omega \to \infty} \angle G(j\omega) = \lim_{\omega \to \infty} \angle -\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 = -180^\circ$$

Step 4 : Separating the real and imaginary parts of $G(j\omega)$

$$G(j\omega) = \frac{K}{(1+j\omega T_1)(1+j\omega T_2)} \cdot \frac{(1-j\omega T_1)(1-j\omega T_2)}{(1-j\omega T_1)(1-j\omega T_2)}$$

$$G(j\omega) = \frac{K(1-\omega^2 T_1 T_2)}{1+\omega^2 T_1^2+\omega^2 T_2^2+\omega^4 T_1^2 T_2^2} - j\frac{K\omega(T_1+T_2)}{1+\omega^2 T_1^2+\omega^2 T_2^2+\omega^4 T_1^2 T_2^2}$$

Equating the real part to zero.

$$\frac{K(1-\omega^2 T_1 T_2)}{1+\omega^2 T_1^2+\omega^2 T_2^2+\omega^4 T_1 T_2} = 0$$

$$\therefore \quad \omega^2 = \frac{1}{T_1 T_2} \quad \text{or} \quad \omega = \frac{1}{\sqrt{T_1 T_2}} \quad \& \quad \omega = \pm\infty$$

The frequency at which plot intersects the imaginary axis is $\frac{1}{\sqrt{T_1 T_2}}$.

For positive values of frequencies the polar plot intersects the imaginary axis at $\omega = \infty$

$$\omega = \frac{1}{\sqrt{T_1 T_2}} \quad \text{and} \quad \omega = \infty$$

Value of $G(j\omega)$ when

$$\omega = \frac{1}{\sqrt{T_1 T_2}}$$

$$G(j\omega) = 0 - j\frac{K\frac{1}{\sqrt{T_1 T_2}}(T_1+T_2)}{1+\frac{1}{T_1 T_2}T_2^2+\frac{1}{T_1 T_2}T_1^2+\frac{1}{T_1^2 T_2^2}\cdot T_1^2 T_2^2}$$

$$= -j\frac{K\frac{T_1+T_2}{\sqrt{T_1 T_2}}}{2+\frac{T_2}{T_1}+\frac{T_1}{T_2}} = -j\frac{K\frac{T_1+T_2}{\sqrt{T_1 T_2}}}{\frac{(T_1+T_2)^2}{T_1 T_2}} = -j\frac{K\sqrt{T_1 T_2}}{T_1+T_2}$$

$$|G(j\omega)| = \frac{K\sqrt{T_1 T_2}}{T_1+T_2} \quad \text{and} \quad \angle G(j\omega) = -90^\circ$$

$\therefore$ When $\quad \omega = \frac{1}{\sqrt{T_1 T_2}} \qquad G(j\omega) = \frac{K\sqrt{T_1 T_2}}{T_1+T_2} \angle -90^\circ$

$\qquad \omega = \infty \qquad G(j\omega) = 0 \angle -180^\circ$

Step 5 : Equating the imaginary part to zero

$$\frac{K\omega(T_1+T_2)}{1+\omega^2 T_1^2+\omega^2 T_2^2+\omega^4 T_1 T_2} = 0$$

Equate the imaginary part equal to zero

$$\frac{K\omega^2 T_1 T_2 - K}{\omega + \omega^3 (T_1{}^2 + T_2{}^2 + \omega^2 T_1{}^2 T_2{}^2)} = 0$$

$$\therefore \qquad \omega = \frac{1}{\sqrt{T_1 T_2}} = \frac{1}{\sqrt{T_1 T_2}} \ \& \ \omega = \pm \infty$$

The frequency at the point of intersection on real axis is $\frac{1}{\sqrt{T_1 T_2}}$. Now calculate the value of $G(j\omega)$ at this point.

Put $\qquad \omega = \frac{1}{\sqrt{T_1 T_2}}$ in equation (A)

$$G(j\omega) = -K\frac{T_1 T_2}{T_1 + T_2} \qquad \angle G(j\omega) = 0^\circ$$

$$G(j\omega) = \infty \qquad \angle G(j\omega) = 0^\circ$$

Step 5 : Equate the real part to zero

$$\frac{-\omega K(T_1 + T_2)}{\omega + \omega^3 (T_1{}^2 + T_2{}^2 + \omega^2 T_1 T_2)} = 0$$

$$\therefore \qquad \omega = \infty$$

For positive values of frequencies the polar plot intersects the imaginary axis at $\omega = \infty$

$$\therefore \qquad G(j\omega) = 0 \ \angle -270^\circ$$

Polar plot is shown in Fig. 4.2.

From the polar plot it is clear that in type one system the $j\omega$ term in denominator contributes -90° to the total phase angle. At $\omega = 0$, the magnitude is infinity and phase angle -90°. At $\omega = \infty$, the magnitude becomes zero and curve converges to origin. At low frequency, the polar plot is asymptotic to a line parallel to negative imaginary axis.

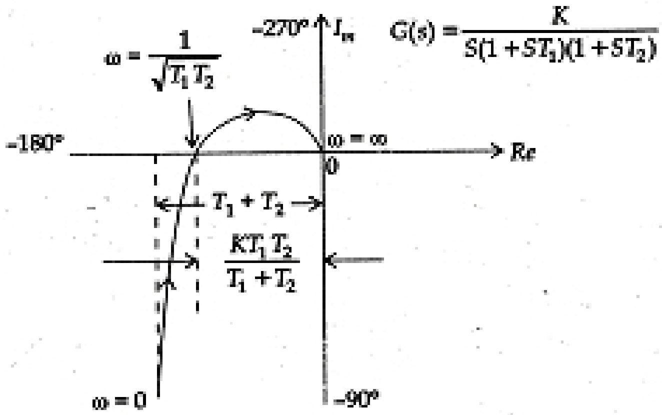


Fig 4.2.

3. TYPE 'TWO' SYSTEM

$$G(s) = \frac{K}{s^2(1 + sT_1)}$$

Put $s = j\omega$

$$G(j\omega) = \frac{K}{(j\omega)^2\,(1 + j\omega T_1)} = \frac{K}{-\omega^2\sqrt{1 + (\omega T_1)^2}} \ \angle -180^\circ - \tan^{-1}\omega T_1$$

$$\lim_{\omega \to 0} |G(j\omega)| = \lim_{\omega \to 0} \frac{K}{-\omega^2\sqrt{1 + (\omega T_1)^2}} = \infty$$

$\therefore \qquad \omega = 0$ and $\omega = \pm\infty$

When $\quad \omega = 0 \qquad |G(j\omega)| = K \qquad \angle G(j\omega) = 0°$

$\omega = \infty \qquad |G(j\omega)| = 0 \qquad \angle G(j\omega) = 0°$

The coordinates of the intersection points '*A*' and '0'

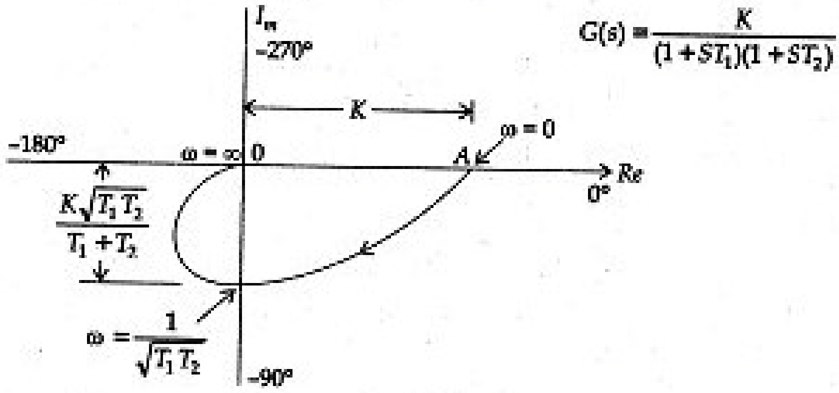


Fig. 4.1.

2. TYPE 'ONE' SYSTEM

$$G(s) = \frac{K}{S(1+ST_1)(1+ST_2)}$$

Step 1 : Put $\quad s = j\omega$

$$G(j\omega) = \frac{K}{j\omega(1+j\omega T_1)(1+(j\omega T_2)}$$

$$G(j\omega) = \frac{K}{\omega\sqrt{1+(\omega T_1)^2}\ \sqrt{1+(\omega T_2)^2}} \angle -90° - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

Step 2 : Taking the limit for the magnitude of $G(j\omega)$

$$\lim_{\omega \to 0} |G(j\omega)| = \lim_{\omega \to 0} \frac{K}{\omega\sqrt{1+(\omega T_1)^2}\ \sqrt{1+(\omega T_2{}^2)}} = \infty$$

$$\lim_{\omega \to \infty} |G(j\omega)| = \lim_{\omega \to \infty} \frac{K}{\omega\sqrt{1+(\omega T_1)^2}\ \sqrt{1+(\omega T_2)^2}} = 0$$

Step 3 : Taking the limit for the phase angle of $G(j\omega)$

$$\lim_{\omega \to 0} \angle G(j\omega) = \lim_{\omega \to 0} \angle -90° - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 = -90°$$

$$\lim_{\omega \to \infty} \angle G(j\omega) = \lim_{\omega \to \infty} \angle -90° - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 = -270°$$

Step 4 : Separating the real and imaginary parts

$$G(j\omega) = \frac{K}{j\omega(1+j\omega T_1)(1+j\omega T_2)}$$

$$= \frac{-\omega K(T_1 + T_2)}{\omega + \omega^3(T_1{}^2 + T_2{}^2 + \omega^2 T_1{}^2 T_2{}^2)} + \frac{j(K\omega^2 T_1 T_2 - K)}{\omega + \omega^3(T_1{}^2 + T_2{}^2 + \omega^2 T_1{}^2 T_2{}^2)} \quad \text{...(A)}$$

Equate the imaginary part equal to zero

$$\frac{K\omega^2 T_1 T_2 - K}{\omega + \omega^3 (T_1^2 + T_2^2 + \omega^2 T_1^2 T_2^2)} = 0$$

$$\therefore \qquad \omega = \frac{1}{\sqrt{T_1 T_2}} = \frac{1}{\sqrt{T_1 T_2}} \ \& \ \omega = \pm\infty$$

The frequency at the point of intersection on real axis is $\frac{1}{\sqrt{T_1 T_2}}$. Now calculate the value of $G(j\omega)$ at this point.

Put $\omega = \frac{1}{\sqrt{T_1 T_2}}$ in equation (A)

$$G(j\omega) = -K\frac{T_1 T_2}{T_1 + T_2} \qquad \angle G(j\omega) = 0^\circ$$

$$G(j\omega) = \infty \qquad \angle G(j\omega) = 0^\circ$$

Step 5 : Equate the real part to zero

$$\frac{-\omega K(T_1 + T_2)}{\omega + \omega^3 (T_1^2 + T_2^2 + \omega^2 T_1 T_2)} = 0$$

$$\therefore \qquad \omega = \infty$$

For positive values of frequencies the polar plot intersects the imaginary axis at $\omega = \infty$

$$\therefore \qquad G(j\omega) = 0 \angle -270^\circ$$

Polar plot is shown in Fig. 4.2.

From the polar plot it is clear that in type one system the $j\omega$ term in denominator contributes -90° to the total phase angle. At $\omega = 0$, the magnitude is infinity and phase angle -90°. At $\omega = \infty$, the magnitude becomes zero and curve converges to origin. At low frequency, the polar plot is asymptotic to a line parallel to negative imaginary axis.

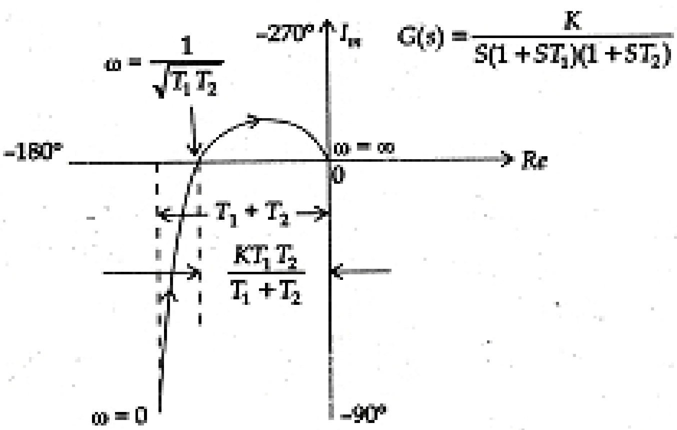


Fig 4.2.

3. TYPE 'TWO' SYSTEM

$$G(s) = \frac{K}{s^2(1 + sT_1)}$$

Put $s = j\omega$

$$G(j\omega) = \frac{K}{(j\omega)^2 (1 + j\omega T_1)} = \frac{K}{-\omega^2 \sqrt{1 + (\omega T_1)^2}} \angle -180^\circ - \tan^{-1}\omega T_1$$

$$\lim_{\omega \to 0} |G(j\omega)| = \lim_{\omega \to 0} \frac{K}{-\omega^2 \sqrt{1 + (\omega T_1)^2}} = \infty$$

$$\lim_{\omega \to \infty} |G(j\omega)| = \lim_{\omega \to \infty} \frac{K}{-\omega^2 \sqrt{1+(\omega T_1)^2}} = 0$$

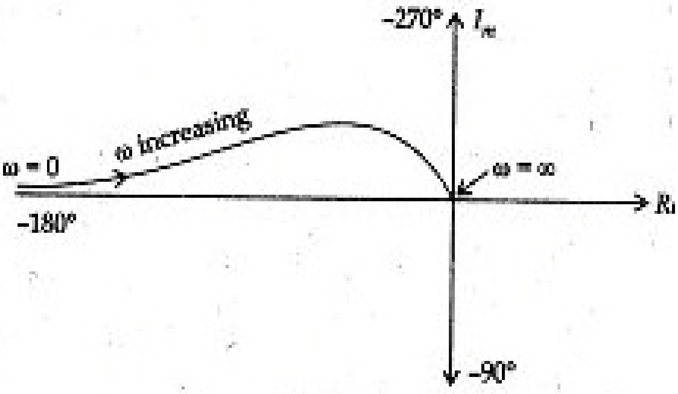
$$\lim_{\omega \to 0} \angle -180° - \tan^{-1} \omega T_1 = -180°$$

$$\lim_{\omega \to \infty} \angle G(j\omega) = \lim_{\omega \to \infty} \angle -180° - \tan^{-1} \omega T_1 = -270°$$

The presence of S^2 in the denominator contributes a constant – 180° to the angle of $G(j\omega)$ for all frequencies.

The polar plot is a smooth curve whose angle decreases continuously from – 180° to 270°. The plot is shown in figure.

From the polar plot it is clear that at $\omega = 0$, magnitude is infinity and phase angle – 180°, at $\omega = \infty$ magnitude is zero and at low frequencies the polar is asymptotic to a line parallel to negative real axis.



BODE PLOT

4.10. BODE PLOT

Bode plot is a graphical representation of the transfer function for determining the stability of the control system. Bode plot consists of two separate plots. One is a plot of the logarithm of the magnitude of a sinusoidal transfer function, the other is a plot of the phase angle, both plots are plotted against the frequency. The curves are drawn on semilog graph paper, using the log scale for frequency and linear scale for magnitude (in decibles) or phase angle (in degrees). The magnitude is represented in decibles. Thus, Bode plot consists of

(i) $20 \log_{10} |G(j\omega)|$ V_s $\log\omega$.

(ii) Phase shift V_s $\log\omega$

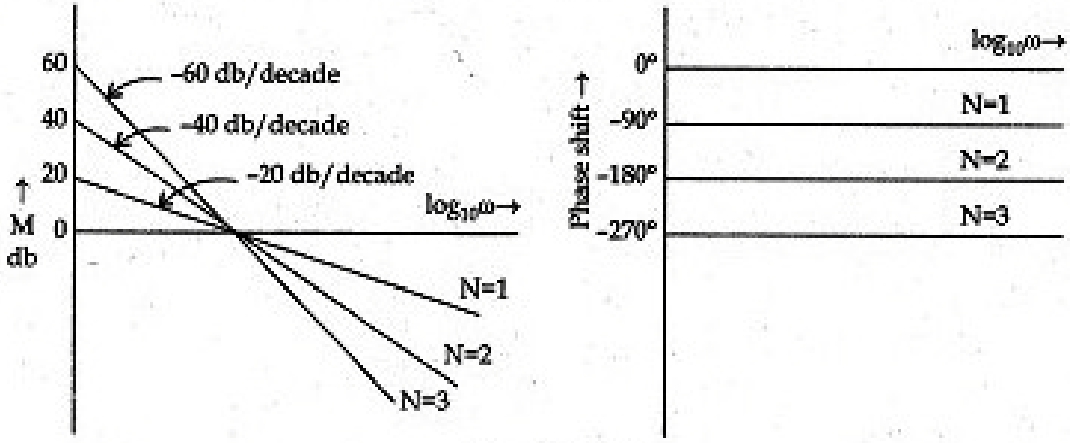


Fig. 4.21.

Case 3 : $G(s) = S$

Put $S = j\omega$

$$G(j\omega) = j\omega$$

$$M = 20\,log_{10} |G(j\omega)| = 20\log_{10}\omega$$

$$\angle G(j\omega) = +\ 90°$$

The plot $M\ V_s\ \log_{10}\omega$ is a straight line having a slope of + 20 db/dec. and angular phase shift of + 90°.

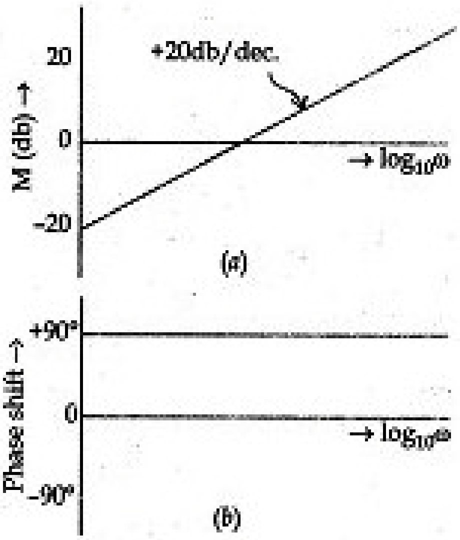


Fig. 4.22.

Case 4 : $G(s) = \dfrac{1}{1+sT}$

Put $s = (j\omega)$

$\therefore$ $G(j\omega) = \dfrac{1}{1+j\omega T}$

$$|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2T^2}}$$

$$20\log_{10}|G(j\omega)| = 20\log_{10}\left[\frac{1}{\sqrt{1+\omega^2T^2}}\right]$$

$$= 20\log_{10}1 - 20\log_{10}\sqrt{1+\omega^2T^2}$$

$$= -20\log_{10}\sqrt{1+\omega^2T^2} \qquad \because \log_{10}1 = 0$$

Put the different values of ω, we will get $|G(j\omega)|$ consider following two cases.

(*a*) For $\omega T << 1$ (very low frequencies)

$$-20\log_{10}\sqrt{1+\omega^2T^2} = -20\log_{10}\sqrt{1} = 0$$

$\therefore$ $M = 0$ for $\omega T << 1$ or $\omega \le \dfrac{1}{T}$

(*b*) For $\omega T >> 1$ (very high frequencies)

$$-20\log_{10}\sqrt{1+\omega^2T^2} = -20\log_{10}\sqrt{\omega^2T^2}$$

$$= -\ 20\log_{10}\omega T \text{ for } \omega >> 1/T$$

The main advantage of using Bode plot is that multiplication of magnitudes can be converted nto addition.

Consider open loop transfer function of a closed loop control system

$$G(s)\,H(s) = \frac{K(1+sT_a)(1+sT_b)....}{s^N(1+sT_1)(1+sT_2)....}$$

Put $$S = j\omega$$

$$G(j\omega)\,H(j\omega) = \frac{K(1+j\omega T_a)(1+j\omega T_b).......}{(j\omega)^N(1+j\omega T_1)(1+j\omega T_2)......}$$

$$20\log_{10}|G(j\omega)\ H(j\omega)| = \left(20\log K + 20\log\sqrt{1+\omega^2T^2_a} + 20\log\sqrt{1+\omega^2T^2_b}\right)....)$$

$$-\left(20N\log\omega + 20\log\sqrt{1+\omega^2T_1^2} + 20\log\sqrt{1+\omega^2T_2^2}\right)....)$$

Hence, in order to get $|G(j\omega)\,H(j\omega)|$ we will have to obtain the individual plots and adding individual components, the resultant can be obtained. Suppose, $H(s) = 1$.

Case 1. The Gain K

$$G(s) = K$$

Put $$s = j\omega$$

$$G(j\omega) = K$$

$$20\log_{10}|G(j\omega)| = 20\log_{10}K \qquad ...(4.1)$$

$$\text{Phase angle } \phi = \angle G(j\omega) = 0° \qquad ...(4.2)$$

From equations (4.1) and (4.2) it is clear that the magnitude is independent of $\log_{10}\omega$ and phase angle always zero. The plots are shown in Fig. (4.20).

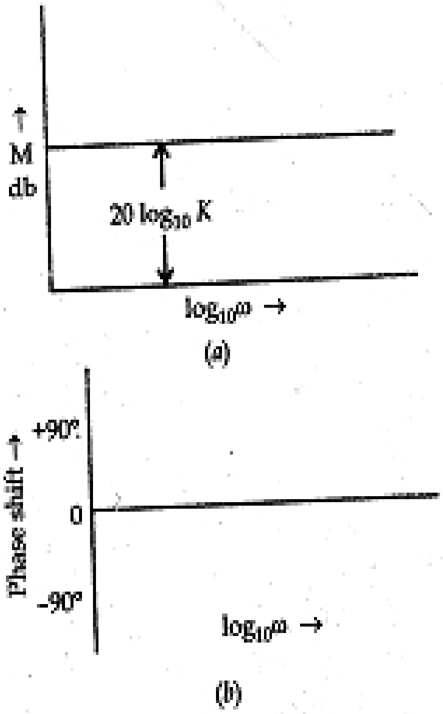


Fig. 4.20.

Case 2 :

$$G(s) = \frac{1}{S^N}$$

Put $$s = (j\omega)^N$$

$$\therefore \quad G(j\omega) = \frac{1}{(j\omega)^N}$$

$$20\log_{10}|G(j\omega)| = 20\log_{10}\frac{1}{(j\omega)^N}$$

$$= 20\log_{10}(j\omega)^{-N}$$

$$= -20\,N\log_{10}(\omega)$$

$$\angle G(j\omega)^N = -90\,N°$$

where $N = 1,2,3,$ ....

The plot M Vs $\log_{10}\omega$ is a straight line. For $N = 1$ the line has a slope of -20 db/decade and angle $-90°$. For $N = 2$, the slope of the line will be -40 db/decade and angle will be $-180°$ and so on.

Hence, M Vs $\log_{10}\omega$ has two parts

(i) One part having $M = 0$ for $\omega << 1/T$

(ii) In other part M varies as a straight line with slope of – 20 db/decade for $\omega >> \frac{1}{T}$

$\omega = \frac{1}{T}$ is called break frequency or corner frequency

$$M = -20\log_{10}\omega T = -20(\log_{10}\omega + \log_{10}T)$$

$$M = -20\log_{10}\omega - 20\log_{10}T$$

$$= -20\log_{10}\omega + 20\log_{10}1/T \quad ...(4.3)$$

The above two parts of the graph intersect 0 *db* axis is determined by equating the eq^n (4.3) to zero

$$0 = -20\log_{10}\omega + 20\log_{10}1/T$$

$\therefore$ $\omega = 1/T$ is called break frequency.

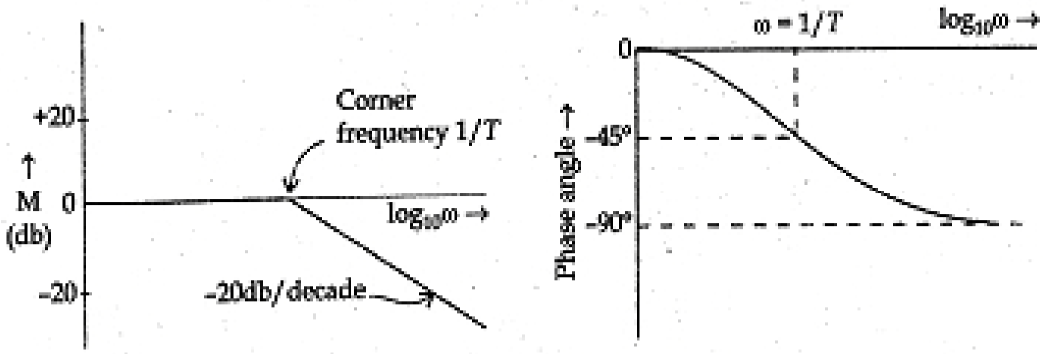


Fig. 4.23.

Case 5 : $G(s) = (1 + sT)$

Put $s = j\omega$

$$G(j\omega) = (1 + j\omega T)$$

$$|G(j\omega)| = \sqrt{1+\omega^2T^2}$$

$$20\log_{10}|G(j\omega)| = 20\log_{10}\sqrt{1+\omega^2T^2}$$

(i) When $\omega T << 1$

$$M = 20\log_{10}\sqrt{1} = 0\ db$$

(ii) When $\omega T >> 1$

$$M = 20\log_{10}\omega T$$

$$M = 20\log_{10}\omega T = 20\log_{10}\frac{\omega}{1/T}$$

$$= 20\log_{10}\omega - 20\log_{10}\frac{1}{T}$$

Equate the above equation to zero

$$0 = 20\log_{10}\omega - 20\log_{10}\frac{1}{T}$$

$\therefore$ $\omega = \frac{1}{T}$ corner frequency.

Thus, the two parts of the graph intersects the '0' db axis at $\omega = \frac{1}{T}$. The second part is a straight line having the slope of + 20 db/decade.

Phase Angle Plot

$$\phi = \angle G(j\omega) = \tan^{-1} \omega T$$

(*i*) At very low frequencies ωT is very very small

$$\phi = \tan^{-1}(0) = 0°$$

(*ii*) At $\omega T = 1$

$$\phi = \tan^{-1} 1 = 45°$$

(*iii*) At very high frequencies

$$\phi = \tan^{-1}(\infty) = 90°$$

Thus, the value of ϕ gradually changes from 0° to 90° as ω increases from 0 to very high values.

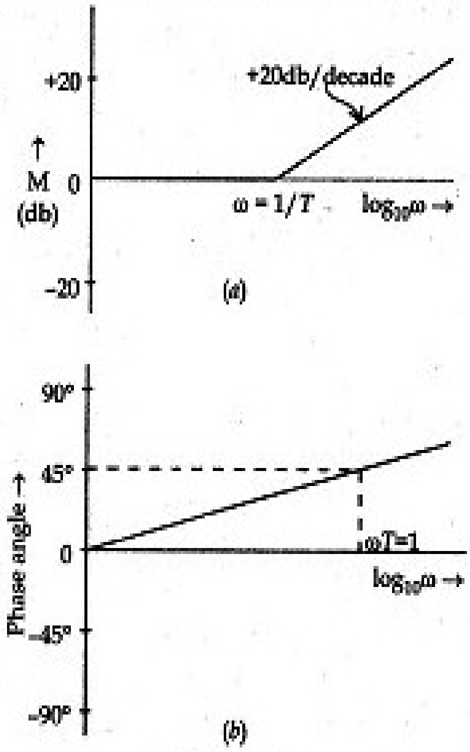


Fig. 4.24.

Case 6 : General second order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Put $s = j\omega$

$$G(s) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} = \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n\omega + \omega_n^2}$$

$$G(s) = \frac{w_n^2}{\omega_n^2 - \omega^2 + j2\zeta\omega_n\omega} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\frac{\omega}{\omega_n}}$$

$$20\log_{10}|G(j\omega)| = 20\log_{10}\left|\frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\frac{\omega}{\omega_n}}\right| = -20\log_{10}\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}$$

Suppose $\frac{\omega}{\omega_n} = u$

$\therefore \quad 20\log_{10}|G(j\omega)| = M = -20\log_{10}\sqrt{(1-u^2)^2 + 4\zeta^2 u^2}$

Consider the two cases

1. $u << 1 \quad i.e \frac{\omega}{\omega_n} << 1$

$$M = -20\log_{10}\sqrt{1} = 0 \text{ db.}$$

2. $u >> 1 \; i.e \frac{\omega}{\omega_n} >> 1$

$$M = -20\log_{10}\sqrt{(u^2)^2} = -20\log_{10} u^2 = -40\log_{10} u$$

So, it is a straight line having slope of – 40 db/dec. and passing through the point u.

Therefore, the asymptotic plot consists of

(i) $M = 0$ $\qquad u << 1$

(ii) $M = -40 \log_{10} u$ $\qquad u >> 1$

'hase Angle Plot

$$\phi = \angle G(j\omega) = -\tan^{-1}\frac{2\xi u}{1-u^2}$$

(i) For small value of u, u^2 is small

$$\therefore \qquad \phi = -\tan^{-1} 2\xi u$$

(ii) For large value of u, $u^2 >> 1$

$$\therefore \qquad \phi = +\tan^{-1}\frac{2\zeta}{u}$$

(iii) When $u = 1$

$\phi = -\tan^{-1}\infty = -90°$

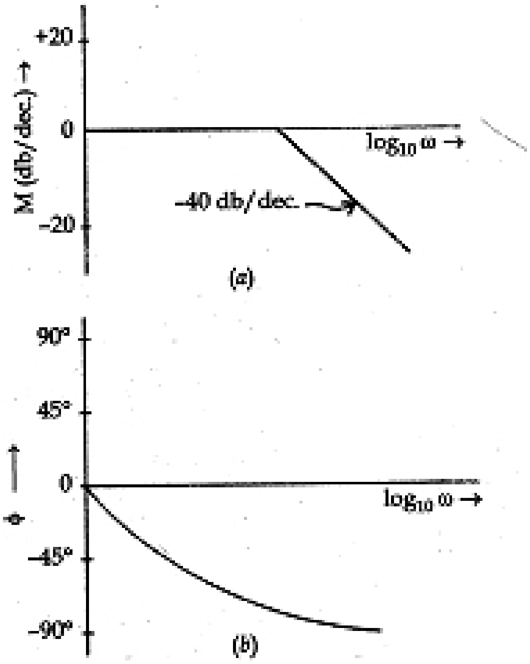


Fig. 4.25.

nitial Slope of Bode Plot

Let $\qquad G(s)\,H(s) = \dfrac{K}{S^N}$

Put $\qquad s = j\omega$

$$G(j\omega)\,H(j\omega) = \frac{K}{(j\omega)^N}$$

$$20 \log_{10} |G(j\omega)\,H(j\omega)| = 20\log_{10}\left|\frac{K}{(jw)^N}\right| = 20 \log_{10} K - 20\,N \log_{10}\omega \qquad ...(4.4)$$

1. For $N = 0$ (Type zero system)

$20 \log_0 |G(j\omega)\,H(j\omega)| = 20 \log_{10} K.$

This is a straight line. The graph is shown in Fig. 4.26.

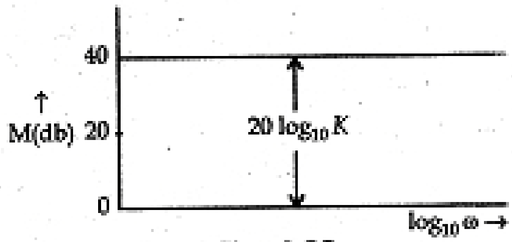


Fig. 4.26.

2. For $N = 1$ (type one system)

Put $N = 1$ in equation (4.4)

$20\log_{10} |G(j\omega)\,H(j\omega)| = 20 \log_{10} K - 20 \log_{10}\omega$

Intersection with 0 db axis

$$0 = 20 \log_{10} K - 20 \log_{10}\omega$$

$$\therefore \qquad K = \omega$$

locate $\omega = K$ on 0 db axis and at this point draw a line of – 20 db/decade produce it till it intersect the y-axis that will be the starting point on Bode plot.

3. For $N = 2$ (type two system)

Put $N = 2$ in equation (4.4)

$$20 \log_{10} |G(j\omega)\,H(j\omega)| = 20 \log_{10} K - 20.\,2 \log_{10}\omega$$

$$= 20 \log_{10} K - 40 \log_{10}\omega$$

Intersection with 0 db axis

$$0 = 20 \log_{10} K - 40 \log_{10}\omega$$

$$20 \log_{10} K = 40 \log_{10} \omega$$
$$20 \log_{10} K = 20 \log_{10} \omega^2$$
$$\omega^2 = K$$
$$\omega = \sqrt{K}$$

Hence, graph intersect the 0 db axis at $\omega = \sqrt{K}$. Locate $\omega = \sqrt{K}$ on 0 db axis and draw a line −40 db/dec. and produce it to the y-axis. Graph having the slope of −40 db/decade is shown in Fig. 4.27.

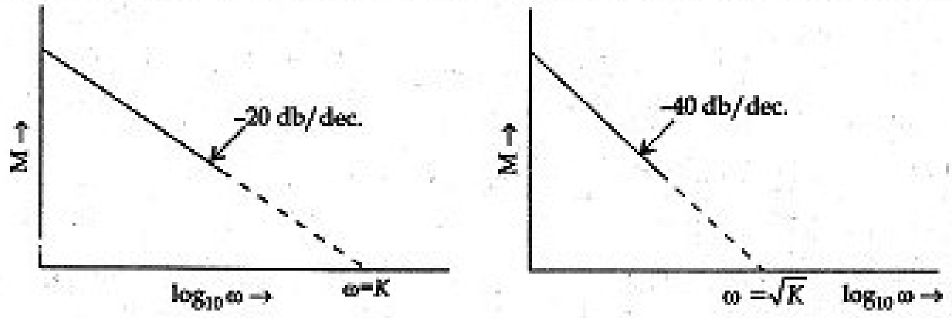


Fig. 4.27.

Table 4.3.

Type of the System N	Initial Slope 0db Axis	Intesection with
0	0 db/decade	Parallel to 0 axis
1	− 20 db/dec.	$= K$
2	− 40 db/dec.	$= \sqrt{K}$
3	− 60 db/dec.	$= K^{1/3}$
⋮	⋮	⋮
N	−20N db/dec.	$K^{1/N}$

4.12. PROCEDURE FOR DRAWING THE BODE PLOTS

Consider the transfer function

$$G(s) = \frac{K(1+sT_a)(1+sT_b)\,.......}{S^N(1+sT_1)(1+sT_2)\,.......\left[1+2\zeta\left(\frac{\omega}{\omega_n}\right)+\left(j\frac{\omega}{\omega_n}\right)^2\right]} \quad ...(4.7)$$

where N is the number of poles at the origin *i.e.* N defines the type of system.

For type zero system $K = K_p$

For type one system $K = K_v$

For type two system $K = K_a$

In above transfer function put $s = j\omega$

$$G(j\omega) = \frac{K(1+j\omega T_a)(1+j\omega T_b).....}{(j\omega)^N(1+j\omega T_1)(1+j\omega T_2).....[1+2\zeta(\omega/\omega_n)+(j\omega/\omega_n)^2]} \quad ...(4.8)$$

$$20\log_{10}|G(j\omega)| = 20\log K + 20\log\sqrt{1+\omega^2 T_a^2} + 20\log\sqrt{1+\omega^2 T_b^2} + ...-20N\log\omega$$

$$-20\log\sqrt{1+\omega^2 T_1^2} - 20\log\sqrt{1+\omega^2 T_2^2} - 20\log\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\xi^2\left(\frac{\omega}{\omega_n}\right)^2} \quad ...(4.9)$$

Phase angle

$$\angle G(j\omega) = \tan^{-1}\omega T_a + \tan^{-1}\omega T_b + ...\, N(90°) - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2\, ...\, \tan^{-1}\left[\frac{2\zeta\omega\omega_n}{\omega^2_n - \omega^2}\right] \quad ...(4.10)$$

Step 1 : I dentify the corner frequency.

Step 2 : Draw the asymptotic magnitude plot. The slope will change at each corner frequency by + 20 db / dec. for zero and – 20 db / dec for pole. For complex conjugate pole and zero the slope will change by $\mp$ 40 db/ decade.

Step 3 : (*i*) For type zero system draw a line upto first (lowest) corner frequency having 0 db dec. slope.

(*ii*) For type one system draw a line having slope – 20 db/ dec. upto $\omega = K$. Mark first (lowest) corner frequency.

(*iii*) For type two system draw the line having slope – 40 db/ dec. upto $\omega = \sqrt{K}$ and so on. Mark first corner frequency.

Step 4 : Draw a line upto second corner frequency by adding the slope of next pole or zero to the previous slope and so on.

Step 5 : Calculate phase angle for different values of ω from the equation (4.10) and join all points.

4.13. PHASE MARGIN & GAIN MARGIN

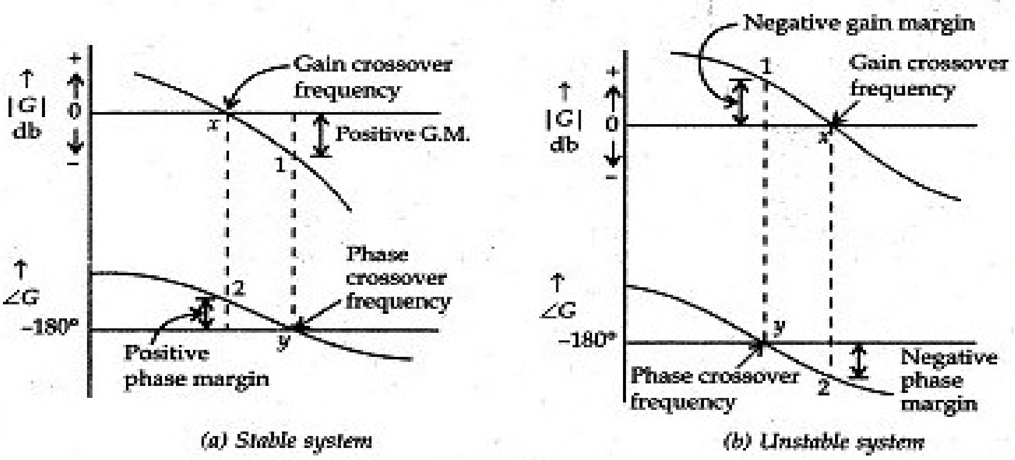


Fig. 4.29.

Positive gain margin means the system is stable and negative gain margin means the system is unstable. For minimum phase system both phase margin and gain margin must be positive for the system to be stable.

The point at which the magnitude curve crosses the 0 db line is the gain crossover frequency. The phase crossover frequency is the point where the phase curve crosses the 180° line.

Gain Margin : Gain margin is defined as the margin in gain allowable by which gain can be increased till system reaches on the verge of instability. Mathematically gain margin is defined as the reciprocal of the magnitude of the $G(j\omega)\ H(j\omega)$ at phase cross over frequency.

$$\therefore \quad \text{G.M.} = \frac{1}{|G(j\omega)\ H(j\omega)|_{\omega=\omega_{c_2}}}$$

where ω_{c_2} = phase crossover frequency.

Generally, G.M. is expressed in decibels

$$\therefore \text{ In decibels} \quad \text{G.M.} = 20 \log \frac{1}{|G(j\omega)\ H(j\omega)|_{\omega=\omega_{c_2}}}$$

$$\text{or} \quad \text{G.M.} = -20 \log_{10} |G(j\omega)\ H(j\omega)|_{\omega=\omega_{c_2}}$$

Phase Margin : For gain the additional phase lag can be introduced without affecting the magnitude plot. Therefore, phase margin can be defined as the amount of additional phase lag which can be introduced in the system till system reaches on the verge of instability is called as phase margin (P.M.). Mathematically phase margin can be defined as

$$\text{P.M.} = \left[\angle G(j\omega)\ H(j\omega)|_{\omega=\omega_{c_1}}\right] - (-180°)$$

$$\text{P.M.} = 180° + \angle G(j\omega)\ H(j\omega)|_{\omega=\omega_{c_1}}$$

where ω_{c_1} = Gain crossover frequency.

5.15. NYQUIST CRITERION

The characteristic equation is given by

$$D(s) = 1 + G(s)H(s) \qquad ...(5.24)$$

The zeros of $D(s)$ are the roots of the characteristic equation. For a feedback system the necessary and sufficient condition is that all zeros of $1 + G(s)H(s)$ that is the roots of the characteristic equation must have negative real part *i.e.*, they must lie in the left half of s-plane. In order to determine the presence of zeros in right half of s-plane we choose a contour as shown in Fig. 5.38 called Nyquist contour. Let there are 'Z' zeros and 'P' poles in the right half of s-plane . If this contour is mapped in $D(s)$ plane as Γ_D then Γ_D enclosesthe origin N times (where $N = Z\text{-}P$) in clockwise. Hence the system is unstable because the clockwise encirclement is possible only when there are zeros of $D(s)$ in right half of s-plane.

A feedback system (close loop system) is stable if and only if there is no zeros of $D(s)$ in the right half of s-plane. *i.e.* $Z = 0$

$$\therefore \qquad N = -P$$

Therefore, for a closed loop system to be stable, the number of counter clockwise encirclement of the origin of $D(s)$ plane by Γ_D should equal the number of right half s-plane poles of $D(s)$ which are the poles of open loop transfer function $G(s)H(s)$.

Since $\qquad D(s) = 1 + G(s)H(s)$

or $\qquad G(s)H(s) = D(s) - 1$

The contour Γ_D in $D(s)$ plane can be mapped in $G(s)H(s)$ plane. Γ_{GH} by shifting horizontally to the left by one unit. Thus the encirclement of the origin by the contour Γ_D is equivalent to the encirclement of the point $(-1+j0)$ by the contour Γ_{GH} as shown in Fig. 5.39.

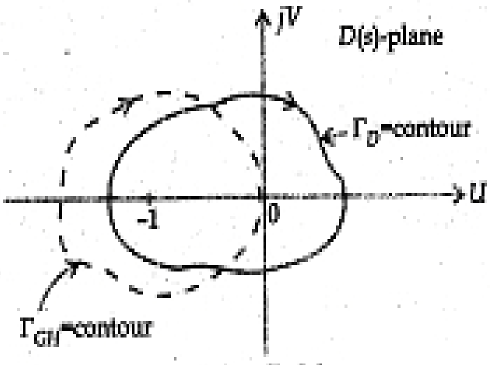


Fig. 5.41.

In most single loop feedback system $G(s)H(s)$ has no poles in the right half plane *i.e*, $P = 0$ then closed loop system is stable if $N = P = 0$.

So, we can say that A closed loop system with $P = 0$ is stable if the net encirclement of the origin of $D(s)$ plane by Γ_D contour is zero.

Now, we can state the Nyquist stability criterion as follows:

A feedback system or closed loop system is stable if the contour Γ_{GH} of the open loop transfer function $G(s)H(s)$ corresponding to the Nyquist contour in the s-plane encircles the point $(-1 + j0)$ in counterclockwise direction and the number of counterclockwise encirclements about the $(-1 + j0)$ equals the number of poles of $G(s)H(s)$ in the right half of s-plane *i.e.*, with positive real parts.

In common case of open loop stable system, the closed loop system is stable if the contour Γ_{GH} of $G(s)H(s)$ does not pass through or does not encircle $(-1 + j0)$ point, *i.e.*, net encirclement is zero.

5.16. GENERAL CONSTRUCTION RULES OF THE NYQUIST PATH

Consider the Fig. 5.37

Table 5.1.

Path *ab*	$s = j\omega$	$0 < \omega < \omega_o$	...(5.25)
Path *bc*	$s = \lim_{P \to 0} (j\omega_o + Pe^{j\theta})$	$-90° \le \theta \le 90°$	...(5.26)
Path *c d*	$s = j\omega$	$\omega_o \le \omega \le \infty$	...(5.27)
Path *def*	$s = \lim_{R \to \infty} Re^{j\theta}$	$-90° \le \theta \le 90°$	...(5.28)
Path *fg*	$s = j\omega$	$-\infty < \omega < -\omega_o$	...(5.29)
Path *gh*	$s = \lim_{P \to 0} (j\omega_o + Pe^{j\theta})$	$-90° \le \theta \le 90°$	...(5.30)
Path *hi*	$s = j\omega$	$-\omega_o \le \omega \le 0$	...(5.31)
Path *ija*	$s = \lim_{P \to 0} Pe^{j\theta}$	$-90° \le \theta \le 90°$	...(5.32)

Step 1 : Check $G(s)$ for poles on $j\omega$ axis and at the origin.

Step 2 : Using equation (5.25) to equation (5.27) sketch the image of the path $a - d$ in the $G(s)$-plane. If there are no poles on $j\omega$ axis equation (5.26) need not be employed.

Step 3 : Draw the mirror image about the real axis of the sketch resulting from step 2.

Step 4 : Use equation (5.28) plot the image of path def. This path at infinity usually plot into a point in the $G(s)$ -plane.

Step 5 : Use equation (5.32) plot the image of path *ija* (pole at origin)

Step 6 : Connect all curves drawn into the previous steps.

EXAMPLE 5.44. **Determine the closed loop stability of a control system whose open loop transfer function is**

$$G(s)\, H(s) = \frac{K}{s\,(1+sT)} \qquad \text{(Type '1' system)}$$

Solution : Given that

$$G(s)\, H(s) = \frac{K}{s\,(1+sT)}$$

Put $s = j\omega$

$$G(j\omega)\, H(j\omega) = \frac{K}{j\omega(1+j\omega T)} \qquad ...(5.33)$$

Rationalizing the equation (5.33) and separating into real and imaginary parts.

$$G(j\omega)\, H(j\omega) = -\frac{KT}{1+\omega^2 T^2} - j\frac{K}{\omega(1+\omega^2 T^2)} \qquad ...(5.34)$$

$$\lim_{\omega \to 0} |G(j\omega)H(j\omega)| = \infty$$

$$\lim_{\omega \to 0} \angle G(j\omega)H(j\omega) = -90°$$

$$\lim_{\omega \to \infty} |G(jw)H(jw)| = 0$$

$$\lim_{\omega \to \infty} \angle G(j\omega)H(j\omega) = -180°$$

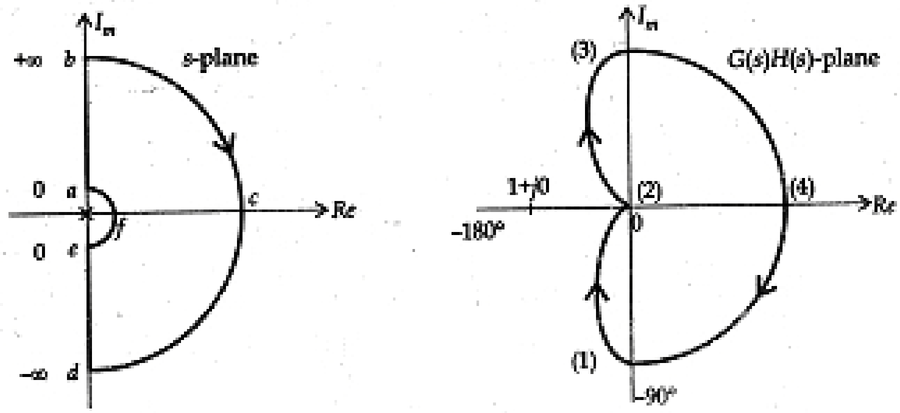


Fig. 5.42. Nyquist plot

The polar plot will lie in third quadrant.

The Nyquist plot is shown in Fig. 5.42. The part for $0 < \omega < +\infty$ is drawn (1) (2) and for $-\infty < \omega < 0$ is shown by the point (2), (3) which is the mirror image of (1), (2). The semicircular detour around the origin in s-plane is mapped into a semicircular path of infinite radius representing a change of phase from $+\pi/2$ to $-\pi/2$.

As the point $(-1 + j0)$ is not encircled by the plot, $N = 0$

$$N = 0 \qquad\qquad P = 0$$

$$\because \quad N = Z - P \qquad\qquad \therefore \quad Z = 0$$

The number of zeros or roots of the characteristic equation with positive real part is nil and hence the closed loop system is stable.

EXAMPLE 5.45. Sketch the Nyquist plot and determine the stability of a unity feedback control system.

$$G(s) = \frac{K}{(1+sT_1)(1+sT_2)} \qquad \text{(Type O system)}$$

Solution : Given that :

$$G(s)\,H(s) = \frac{K}{(1+sT_1)(1+sT_2)}$$

Put $s = j\omega$

$$G(j\omega)\,H(j\omega) = \frac{K}{(1+j\omega T_1)(1+j\omega T_2)} \qquad \text{...(5.35)}$$

$$|G(j\omega)\,H(j\omega)| = \frac{K}{\sqrt{1+\omega^2 T^2{}_1}\,\sqrt{1+\omega^2 T_2{}^2}} \qquad \text{...(5.36)}$$

$$\angle G(j\omega)H(j\omega) = -\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 \qquad \text{...(5.37)}$$

$$\lim_{\omega \to 0} |G(j(\omega)H(j\omega)| = \text{K}$$

$$\lim_{\omega \to 0} \angle G(j\omega)H(j\omega) = 0$$

$$\lim_{\omega \to \infty} |G(j\omega)H(j\omega)| = 0$$

$$\lim_{\omega \to \infty} \angle G(j\omega)H(j\omega) = -180°$$

Rationalize the equation (5.35) and separate the real and imaginary parts.

$$\frac{K}{(1+j\omega T_1)(1+j\omega T_2)} = \frac{K(1-\omega^2 T_1 T_2)}{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)} - j\frac{\omega(T_1+T_2)K}{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)} \quad ...(5.38)$$

Equate the real part to zero, we get

$$\omega = \frac{1}{\sqrt{T_1 T_2}}$$

$$|G(j\omega)H(j\omega)|_{\omega=\frac{1}{\sqrt{T_1T_2}}} = \frac{KT_1T_2}{T_1+T_2}$$

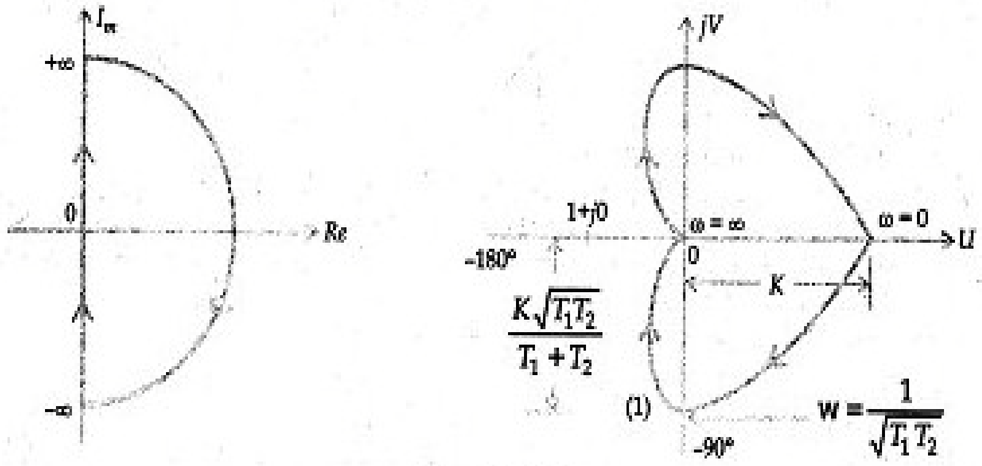


Fig. 5.43.

The plot of $G(j\omega)\,H(j\omega)$ is shown in Fig. 5.43. The infinite semicircular arc of the Nyquist contour maps into origin. As the point $(-1 + jo)$ is not encircled by the plot

$\therefore \quad N = 0$

$P = 0$

$\therefore \quad Z = 0$

Hence, the system is stable.

EXAMPLE 5.46. Using Nyquist criterion, determine the stability of the feedback system which has the following open loop transfer function.

$$G(s)\,H(s) = \frac{K}{s^2(1+sT)} \qquad \textit{(Type '2' system)}$$

Solution : Given that

$$G(s)\,H(s) = \frac{K}{s^2(1+sT)}$$

Put $\quad s = j\omega$

$$G(j\omega)\,H(j\omega) = \frac{K}{(j\omega)^2(1+j\omega T)} \quad ...(5.39)$$

Rationalizing the equation (5.39) and separating the real and imaginary part

$$G(j\omega)\,H(j\omega) = \frac{K}{-\omega^2(1+\omega^2T^2)} + j\frac{K}{\omega(1+\omega^2T^2)} \quad ...(5.40)$$

The Nyquist diagram is shown in the Fig. (5.42). Because of the double pole at $s = 0$, a small semicircular detour at the origin should be made.

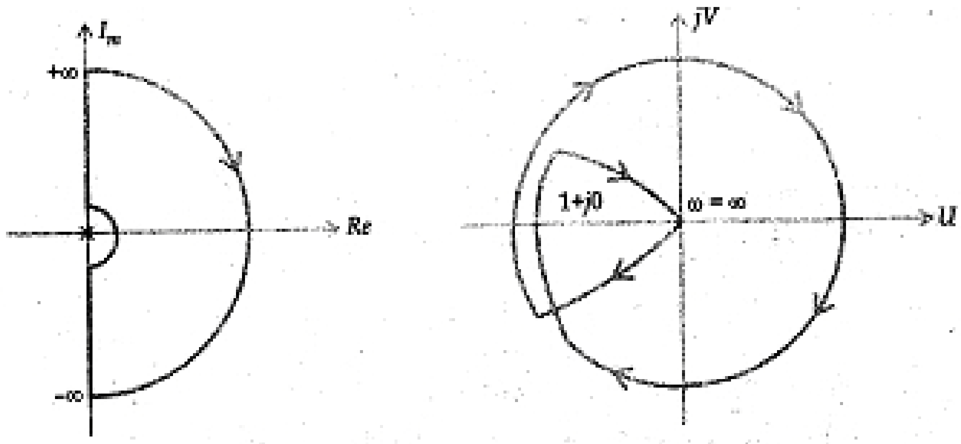


Fig. 5.44.

The point $(-1 + j0)$ is encircled twice. Hence $N = 2$

$$P = 0$$

$$\therefore \quad Z = 2$$

Hence, the system is unstable.

EXAMPLE 5.47. Use Nyquist criterion, determine whether the closed loop system having the following open loop transfer function is stable or not.

$$G(s)\,H(s) = \frac{1}{s\,(1+2s)\,(1+s)}$$

Solution : Given that

$$G(s)\,H(s) = \frac{1}{s\,(1+2s)\,(1+s)}$$

Put $s = j\omega$

$$G(j\omega)\,H(j\omega) = \frac{1}{j\omega(1+j2\omega)\,(1+j\omega)} \qquad \text{...(5.41)}$$

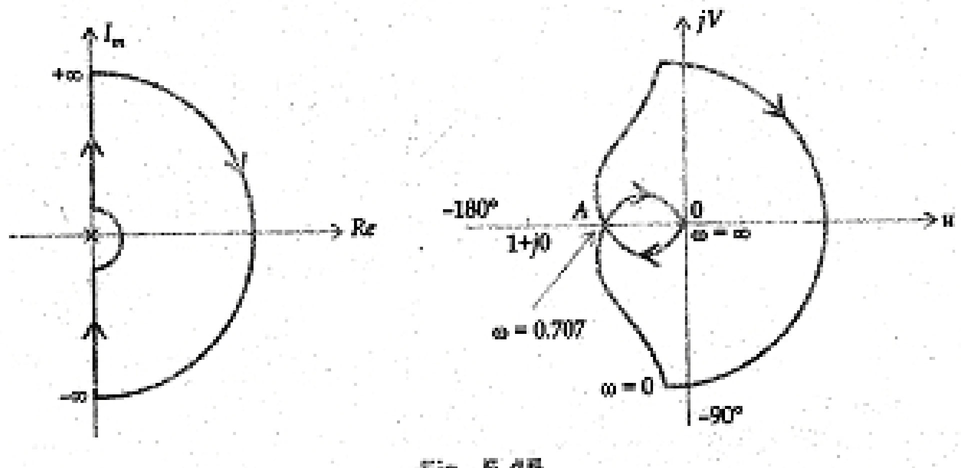


Fig. 5.45.

Rationalizing the equation (5.41) and separate the real and imaginary part.

$$G(j\omega)\,H(j\omega) = \frac{-3}{(1+4\omega^2)\,(1+\omega^2)} - j\,\frac{1-2\omega^2}{\omega(1+4\omega^2)\,(1+\omega^2)} \qquad \text{...(5.42)}$$

CHAPTER-9

STATAE VARIABLE ANALYSIS

8.1. ANALYSIS OF SYSTEMS

The procedure for determining the state of a system is called state variable analysis. The state of a dynamic system is the smallest set of variables such that the knowledge of these variables at $t = t_0$ with the knowledge of the input for $t \geq t_0$ completely determines the behaviour of the system for any time $t \geq t_0$. This set of variables is called state variables.

In earlier chapters we studied the linear system by transfer function, block diagram etc. The transfer function has some drawbacks *e.g.* transfer function is only defined under zero initial conditions and also it is applicable to linear time invariant systems.* Therefore due to these limitation state variable approach is developed. This technique can be used for analysis and design of linear and non-linear, time invariant or time variant and multi input multi-output systems.** The state space analysis involves the description of the system in terms of *I* order differential equations by selecting suitable state variables, the first order derivatives are arranged on left hand side and on right hand side the terms are free from derivatives. The state space techniques have many advantages (Given in next article *i.e.*, 8.2).

8.2. ADVANTAGES OF STATE SPACE TECHNIQUES

This technique has the following advantages.

* If the characteristic of a system does not change with time, then the system is said to be time invariant.

** A system is said to be a single variable system if and only if it has only one input terminal and only one output terminal. A system is said to be multivariable system if and only if it has more than one input terminal or more than one output terminal.

1. This approach can be applied to linear or nonlinear, time variant or time invariant systems.
2. It is easier to apply where the Laplace transform cannot be applied
3. n^{th} order differential equations can be expressed as 'n' equation of first order whose solutions are easier.
4. It is a time domain approach.
5. This method is suitable for digital computer computation because this is a time domain approach.
6. The system can be designed for optimal conditions with respect to given performance indices.

8.3. SOME IMPORTANT DEFINITIONS

State : The state of a system at any time 't_0' is the minimum set of numbers $x_1, x_2, ..., x_n$ which along with the input to the system for time $t \geq t_0$ is sufficient to determine the behaviour of the system for all $t \geq t_0$. In other words, the state of a system represents the minimum amount of information that we need to know about a system at "t_0" such that its future behaviour can be determined without reference to the input before 't_0'. The state can also be defined as the state of a system at time t_0 is the amount of information at t_0, that, together with input $u(t_0, \infty)$ determines the unique behaviour of the system for all $t \geq t_0$. By the behaviour of the system, we mean all responses, including the state of the system. If the system is a network we mean the voltage and current of every branch of the network.

Consider the network shown in Fig. 8.1 if the initial current through the inductor and initial voltage across the capacitor are known, then for any driving voltage the behaviour of the netwrok can be determined. Hence, the inductor current and capacitor voltage can be considered as the state of the network.

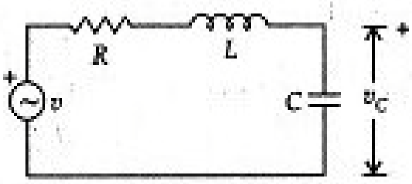


Fig. 8.1.

State Variables : The definition is given in Article 8.1

State Vector : If we need n variables to completely describe the behaviour of a given system, then these n state variables may be considered as n component of a vector x. Such a vector is called state vector, A state vector is thus a vector which determines uniquely the system state $x(t)$ for any time $t \geq t_0$, once the state at $t = t_0$ is given and the input $u(t)$ for $t \geq t_0$ is specified.

State space : The n-dimensional space whose coordinate axes consists of the x_1 axis, x_2 axis ......, x_n axis is called state space. Any state can be represented by a point in the state space.

8.4. STATE SPACE REPRESENTATION

8.4.1. State Space Representation For Electrical Network (Physical Variable Form)

Consider an RLC network shown in Fig. 8.2. Let, the current at time $t = 0$ be $i_L(0)$ and capacitor voltage at time $t = 0$ be $V_c(0)$. Thus, the state of the network at time $t = 0$ is specified by the inductor current and capacitor voltage. Hence, the pair $i_L(0)$, $V_c(0)$ is called the initial state of the network. Similarly at time 't', the pair $i_L(t)$, $V_c(t)$ is called the state of the network at 't'. The variable i_L and V_c are called state variables of the network.

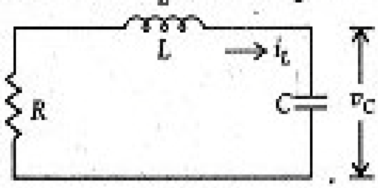


Fig. 8.2.

Apply KVL

$$Ri_L + L\frac{di_L}{dt} + V_c = 0 \qquad ...(8.1)$$

Also, $$i_c = i_L = C\frac{dV_c}{dt}$$ ...(8.2)

From equation (8.1) $$\frac{di_L}{dt} = -\frac{R}{L}i_L - \frac{1}{L}V_c$$ ...(8.3)

$$\frac{dv_c}{dt} = \frac{1}{C}i_L$$ ...(8.4)

Equations of this form are called state equations. In such equations all the variables present are state variables.

Equations (8.3) and (8.4) can be written in matrix form as

$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{dv_c}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i_L \\ V_C \end{bmatrix}$$ ...(8.5)

let $x(t) \Delta \begin{bmatrix} i_L \\ V_C \end{bmatrix}$ and $A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}$

then equation (8.5) can be written as

$$\frac{d}{dt}x(t) = A\,x(t)$$

or $$\dot{x}(t) = A\,x(t)$$

In the linear time-invariant systems, the general form of state equations are

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$ ...(8.6)

$$y(t) = C\,x(t) + D\,u(t)$$ ...(8.7)

These equations are vector differential equations where 'x' is the n-dimensional state vector

y = n-dimensional output vector

u = r-dimensional control vector or input vector

A = $n \times n$ system matrix

B = $n \times r$ control matrix

C = $n \times n$ output matrix

In some cases there is no direct connection between input and output so $D\,u(t)$ will not be there.

$$y(t) = C\,x(t)$$ ...(8.8)

Equations (8.6) and (8.8) can be expressed as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots \\ \vdots \\ a_{n_{n1}} & & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots\cdots & b_{1r} \\ \vdots \\ \vdots \\ b_{n1} & & & b_{nr} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_n \end{bmatrix}$$ ...(8.9)

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & \cdots\cdots & C_{1n} \\ \vdots \\ \vdots \\ C_{m1} & \cdots\cdots & \cdots\cdots & C_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$ ...(8.10)

8.4.2. State Space Representation of *n*th Order Differential Equations

Consider the following examples.

(*a*) For *n*th Order Differential Equation

EXAMPLE 8.1. **A system is described by the differential equation**

$$\frac{d^3y}{dt^3}+6\frac{d^2y}{dt^2}+11\frac{dy}{dt}+10y = 8\,u(t)$$

where y is the output and u is the input to the system. Obtain state space representation of the system. *(R.M.L., University, Faizabad, 2003)*

Solution : Select the state variables as

$$x_1 = y,\; x_2 = \dot{y} \text{ and } x_3 = \ddot{y} \text{ then}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = 8\,u(t) - 10\,x_1 - 11x_2 - 6x_3$$

The last equation is obtained from the given equation.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -10 & -11 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 8\end{bmatrix}u(t) \qquad \text{...(8.11)}$$

Compare equation (8.11) with equation (8.6) we get

$$A = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -10 & -11 & -6\end{bmatrix},\; B = \begin{bmatrix}0\\ 0\\ 8\end{bmatrix},\; x(t) = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

(*b*) State Space Representation of n^{th} Order Linear System with *r* Forcing Function

Consider the following example

EXAMPLE 8.2. **A system is described by the following differential equation. Represent the system in state space.**

$$\frac{d^3x}{dt^3}+3\frac{d^2x}{dt^2}+4\frac{dx}{dt}+4x = u_1(t)+3u_2(t)+4\,u_3(t)$$

and outputs are

$$y_1 = 4\frac{dx}{dt}+3u_1$$

$$y_2 = \frac{d^2x}{dt^2}+4u_2+u_3$$

Solution : Select the state variables as

$$x_1 = x$$

$$\dot{x}_1 = \dot{x} = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = u_1(t) + 3u_2(t) + 4u_3(t) - 3x_3 - 4x_2 - 4x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Outputs :

$$y_1 = 4x_2 + 3u_1$$
$$y_2 = x_3 + 4u_2 + u_3$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

8.4.3. State Space Representation for Transfer Function

Consider the following example

EXAMPLE 8.3. **For the given transfer function, obtain the state model.**

$$G(s) = \frac{y(s)}{u(s)} = \frac{K}{s^3 + a_3 s^2 + a_2 s + a_1}$$

Solution : This transfer function has no zeros.

$$(s^3 + a_3 s^2 + a_2 s + a_1)\, y(s) = Ku(s)$$

or $$s^3 y(s) + a_3 s^2 y(s) + a_2 s\, y(s) + a_1 y(s) = Ku(s)$$

Taking inverse Laplace

$$\dddot{y}(t) + a_3 \ddot{y}(t) + a_2 \dot{y}(t) + a_1 y(t) = K u(t)$$

or $$\dddot{y}(t) = Ku(t) - a_3 \ddot{y}(t) - a_2 \dot{y}(t) - a_1 y(t)$$

Select the state variables as, first state variable as output

$$y(t) = x_1$$
$$\dot{y}(t) = \dot{x}_1 = x_2$$
$$\ddot{y}(t) = \dot{x}_2 = x_3$$
$$\dddot{y}(t) = \dot{x}_3$$

$\therefore$ $$\dddot{y}(t) = -a_3 x_3 - a_2 x_2 - a_1 x_1 + Ku(t)$$

Rewriting the equations

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -a_3 x_3 - a_2 x_2 - a_1 x_1 + Ku(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix} u(t)$$

$$y(t) = [1\ 0\ 0]\, x_1(t)$$

BLOCK DIAGRAM :

The block diagram of the given transfer function is shown in Fig. 8.3.

Now consider another case when the transfer function has zeros.

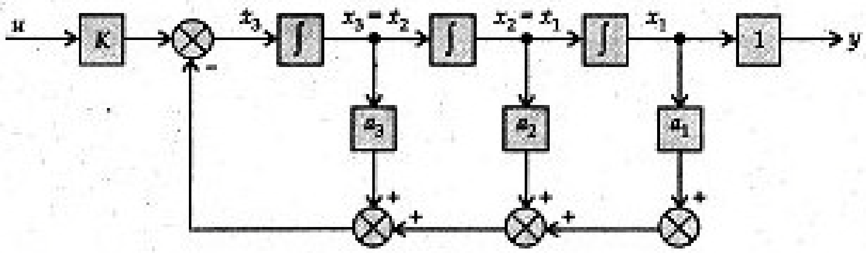


Fig. 8.3.

EXAMPLE 8.4. **Obtain the state model for the given transfer function.**

$$G(s) = \frac{y(s)}{u(s)} = \frac{K(C_2 s + C_1)}{s^3 + a_3 s^2 + a_2 s + a_1}$$

Solution : Break the transfer function in two parts

$$\frac{y(s)}{u(s)} = \frac{x_1(s)}{u(s)} \cdot \frac{y(s)}{x_1(s)} = \frac{K}{s^3 + a_3 s^2 + a_2 s + a_1} (C_2 s + C_1)$$

Now consider $\frac{x_1(s)}{u(s)} = \frac{K}{s^3 + a_3 s^2 + a_2 s + a_1}$

$[s^3 + a_3 s^2 + a_2 s + a_1]\, x_1(s) = Ku(s)$

Taking inverse Laplace

$$\dddot{x}_1(t) + a_3 \ddot{x}_1(t) + a_2 \dot{x}_1(t) + a_1 x_1(t) = Ku(t)$$

$$\dddot{x}_1(t) = -a_3 \ddot{x}_1(t) - a_2 \dot{x}_1(t) - a_1 x_1(t) + Ku(t)$$

Select the state variables as

$$\dot{x}_1 = x_2$$
$$\ddot{x}_1 = \dot{x}_2 = x_3$$
$$\dddot{x}_1 = \dot{x}_3$$

Rewrite

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -a_3 x_3 - a_2 x_2 - a_1 x_1 + Ku$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix} u(t)$$

Consider, $\frac{y(s)}{x_1(s)} = C_2 s + C_1$

$$y(s) = x_1(s)\,[C_2 s + C_1]$$

Take inverse Laplace

$$y(t) = C_1 x_1 + C_2 x_2$$
$$y(t) = [C_1 \;\; C_2 \;\; 0]\, x(t)$$
$$= [C_1 \;\; C_2 \;\; 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

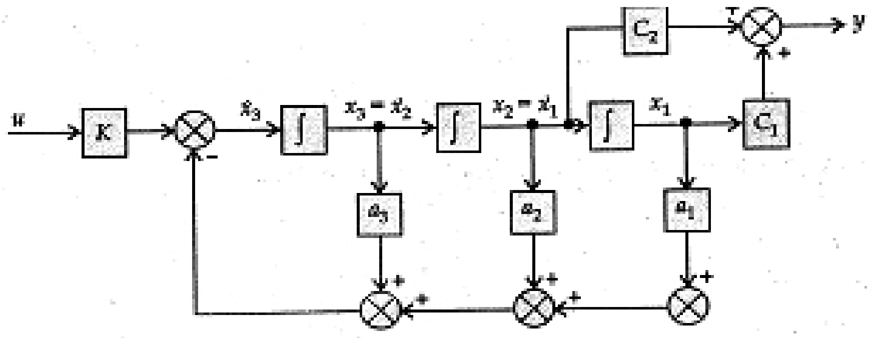


Fig. 8.4.

8.5. SOLUTION OF THE TIME-INVARIANT STATE EQUATION

8.5.1. Solution of Homogeneous State Equation : Laplace Transform Method

We know that

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \text{...(8.12)}$$

$u(t) = 0$ for unforced response

Then $$\dot{x}(t) = Ax(t) \quad \text{...(8.13)}$$

Let us consider the analogous scalar equation

$$\dot{x}(t) = ax(t) \quad \text{...(8.14)}$$

Take the Laplace transform of equation (8.14)

$$sX(s) - x(0) = aX(s)$$

$$(s - a)\, X(s) = x(0)$$

or $$X(s) = (s - a)^{-1}\, x(0) \quad \text{...(8.15)}$$

Take the inverse laplace of equation (8.15)

$$x(t) = e^{at}\, x(0) \quad \text{...(8.16)}$$

If equation (8.16) is the solution of equation (8.14) then the solution of equation (8.13)

$$x(t) = e^{At}\, x(0)$$

$e^{At} = \phi(t)$ = State Transition Matrix (STM)

$$= I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \ldots = \sum_{i=0}^{n} \frac{A^i t^i}{i!} \quad \text{...(8.17)}$$

$$\phi(t) = \pounds^{-1}\phi(s) = \pounds^{-1}[sI - A]^{-1}$$

where $\phi(s)$ = Resolvant matrix

8.5.2. Properties of State Transition Matrices

For time invariant system $\dot{x} = Ax$ and

$$\phi(t) = e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \ldots$$

(*i*) $\phi(0) = e^{A(0)} = I$

(*ii*) $\phi(t) = e^{At} = (e^{-At})^{-1} = [\phi(-t)]^{-1}$